NEW HE-NATURAL HOMOTOPY ANALYSIS TRANSFORM METHOD FOR NONLINEAR DELAY DIFFERENTIAL EQUATIONS

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DEDICATION

I dedicated this thesis to my beloved parents that are always support me with prayers and be in my side in achieving my desired goals. It is also dedicated to my lovely and caring wife Samira Mohammad and my loving Daughter Balkisu Aminu Barde. Lastly to my brothers and sisters.

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ABSTRACT

Delay differential equations (DDEs) are a type of functional differential equations that arise in numerous sciences, engineering, and many other fields of studies. These equations play a vital role in the mathematical modelling of real-life phenomena. Accordingly, systems of these equations provide a significant impact in different settings of applications. Many methods have been used to obtain a solution of various forms of DDEs. However, most of these methods that have been used give the difficulties in finding a convergent approximate analytical solution of nonlinear DDEs. These include divergence of the result as time increases, linearization, restrictive assumptions, and over-dependency of small or large parameters. Therefore, the analytical approximation of nonlinear DDEs has become a challenging task, especially the higher-order system of these equations. In this research, a new analytical method is introduced for solving different classes of nonlinear DDEs. The introduced method is based on the Homotopy analysis method and Natural transform, where the nonlinear terms are simply calculated as a series of He's polynomial. Firstly, this approach is established based on retarded and neutral DDEs with constant and variable coefficients for both proportional and constant delays. The idea is extended to the systems of these types of DDEs, where the He's polynomial is modified to suit the computation of the nonlinear terms of the systems. Secondly, the generalization of the method to the n^{th} order nonlinear single and systems of these equations are provided. In addition, the convergence analysis of each developed algorithm is investigated to guarantee the convergence of the series solution produced by the approach. The established method gives the solution in the form of a rapidly convergent series, leading to the exact or approximate solution of at most three number of iterations of computational terms. Furthermore, unlike some existing methods, the developed method obtains solution without linearization, perturbation, or restrictive assumptions. Solutions to some problems of nonlinear DDEs from real-life applications are obtained to illustrate the effectiveness of the method. Finally, the obtained results are compared to the existing ones as well as the exact solutions that show the method adjusts the interval of convergence for the series solution to avoid round-off of errors and reduces the computational size compared to the reference methods. Hence, the approach is reliable and efficient in solving certain classes of nonlinear problems of DDEs.

ABSTRAK

Persamaan pembezaan lengah (PPL) merupakan satu kelas persamaan pembezaan fungsian yang terdapat dalam banyak bidang sains, kejuruteraan dan lain-lain bidang kajian. Persamaan tersebut memainkan peranan penting dalam pemodelan matematik bagi fenomena kehidupan sebenar. Justeru, sistem persamaan ini memberi kesan yang signifikan dalam aplikasi yang berlainan. Pelbagai kaedah telah dilaksanakan untuk memperoleh penyelesaian kepada pelbagai bentuk PPL. Walau bagaimanapun, kebanyakan kaedah tersebut yang telah digunakan memberi kesukaran untuk mendapatkan penyelesaian analitik anggaran tertumpu bagi PPL tak linear. Ini termasuk penyelesaian yang mencapah apabila masa meningkat, pelinearan, anggapan yang terbatas, dan kebergantungan kepada parameter besar atau kecil. Oleh itu, mendapatkan penyelesaian hampiran analitik bagi PPL tak linear telah menjadi suatu tugas yang mencabar, terutama bagi sistem persamaan peringkat lebih tinggi. Dalam penyelidikan ini, suatu kaedah analitik baharu diperkenalkan untuk menyelesaikan PPL tak linear kelas berbeza. Kaedah yang diperkenalkan ini adalah berdasarkan kepada analisis Homotopy dan jelmaan Natural, iaitu sebutan-sebutan tak linear akan dikira sebagai siri polinomial He. Pertama, pendekatan ini ditubuhkan berdasarkan PPL terencat dan neutral dengan pekali tetap dan boleh berubah untuk kedua-dua lengahan berkadar dan tetap. Idea ini dilanjutkan kepada sistem PPL bagi jenis yang tersebut, di mana polinomial He diubahsuai untuk dipadankan dengan kiraan sebutan tak linear sistem itu. Kedua, pengitlakan kaedah ini ke peringkat n bagi PPL tak linear dan sistem PPL juga diberikan. Tambahan lagi, analisis penumpuan bagi setiap algoritma yang dibangunkan dikaji untuk menjamin penumpuan penyelesaian siri yang dihasilkan daripada pendekatan ini. Kaedah yang ditubuhkan memberi penyelesaian dalam bentuk siri tertumpu secara pantas, yang membawa kepada penyelesaian tepat atau hampir dengan sebutan kiraan yang tidak melebihi tiga lelaran. Tambahan pula, tidak seperti kaedah sedia ada, kaedah yang dibangunkan ini memperoleh penyelesaian tanpa pelinearan, usikan, atau anggapan terbatas. Penyelesaian kepada beberapa masalah PPL tak linear daripada aplikasi kehidupan sebenar diperolehi untuk menunjukkan keberkesanan kaedah tersebut. Akhirnya, hasil yang diperoleh dibandingkan dengan penyelesaian sedia ada dan juga penyelesaian tepat yang menunjukkan bahawa kaedah tersebut dapat melaraskan jarak penumpuan bagi penyelesaian siri untuk mengelakkan ralat pembundaran di samping mengecilkan saiz kiraan berbanding dengan kaedah-kaedah terdahulu. Oleh itu, pendekatan ini boleh dipercayai dan cekap dalam menyelesaikan kelas tertentu bagi masalah PPL tak linear.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

LIST OF ABBREVIATIONS

ADM	-	Adomian Decomposition Method
DDE	-	Delay Differential Equation
DDEs	-	Delay Differential Equations
DE	-	Differential Equation
EMHPM	-	Enhanced Multi-Stage Homotopy Perturbation Method
FDEs	-	Functional Differential Equations
HAM	-	Homotopy Analysis Method
HPM	-	Homotopy Perturbation Method
MHPM	-	Modified Homotopy Perturbation Method
MSVIM	-	Multi-Stage Variational Iteration Method
NT	-	Natural Transform
NDDEs	-	Neutral Delay Differential Equations
ODEs	-	Ordinary Differential Equations
OHAM	-	Optimal Homotopy Asymptotic Method
PDEs	-	Partial Differential Equations
RDDEs	-	Retarded Delay Differential Equations
VIM	-	Variational Iteration Method

LIST OF SYMBOLS

$\mathbb{R}$	-	1-dimentional real Euclidean space
$\lvert.\rvert$	-	Absolute Value
$H(t)$	-	Auxiliary Function
ℓ	-	Auxiliary linear operator
$\hbar$	-	Auxiliary parameter
$\lvert\alpha\rvert$	-	Absolute sum of $\alpha_1, \alpha_2 \ldots \alpha_n$
$[a, b]$	-	Closed Interval on the real line from a to b
$C([a, b], \mathbb{R}^n)$	-	Continuos function mapping the interval $[a, b]$ into $\mathbb{R}^n$
$\times$	-	Cross product
$\in$	-	Element of
$:=$	-	Equivalent to
e	-	Expontial notation
$!$	-	Factorial notation
$'$	-	First derivative
$\frac{d}{dt}$	-	First derivative with respect to t
$\frac{d}{du}$	-	First derivative with respect to u
$\forall$	-	For all
$\phi(t, q)$	-	Function of real variables t and q
$\Gamma(u)$	-	Gamma function of u
$>$	-	Greater than
$\geq$	-	Greater than or equal to
$H_\tau(t)$	-	Heaviside function
I	-	Identity matrix
$\implies$	-	Implies that
∞	-	Infinity
$\int$	-	Integral

$\int_0^t$	-	Integration from 0 to t
$\cap$	-	Intersection
$p(t)$	-	Initial history of delay
$\mathbb{N}^{-}[y(t)]$	-	Inverse Natural transform of the function $y(t)$
$<$	-	Less than
$\leq$	-	Less than or equal to
$L(y)$	-	Linear operator of y
$\mathbb{R}^m$	-	m-dimensional Euclidean Space
$\mathbf{y}_m$	-	m-dimensional iterative vector
$\mathbf{y}$	-	m-dimensiona vector
$\frac{\partial^{m-1}}{\partial q^{m-1}}N[.]$	-	$m-1$-partial derivative of the nonlinear operator N with respect to q
$\frac{\partial^{m-1}}{\partial q^{m-1}}N[.]\|_{q=0}$	-	$m-1$-partial derivative of the nonlinear operator N with respect to q at $q=0$
$y_0^{[m]}(t)$	-	m^{th}-order deformation derivative with respect to t
$H_{i,m}(.)$	-	Modified He's polynomial
$\overrightarrow{y_n}$	-	n-dimensional vector
(n)	-	n^{th} derivative with respect to t
$\frac{d^n}{du^n}$	-	n^{th} derivative with respect to u
$y^{(n_i)}$	-	n_i^{th}-derivative with respect to t
$\mathbb{R}^{n\times n}$	-	$n\times n$-dimensional real Euclidean space
$R_n(s,u)$	-	Natural transform of $y^{(n)}(t)$
$\mathbb{N}^{+}[y(t)]$	-	Natural transform of the function $y(t)$
$F(y)$	-	Nonlinear function of y
N	-	Nonlinear operator
$\\|.\\|$	-	Norm
$(0,1)$	-	Open interval on the real line from 0 to 1
Ψ	-	Open subset in $D\times C$
$\prod_{i=1}^n$	-	Product from index $i=1$ to $i=n$

$R(y)$	-	Remainder of a linear operator
λ	-	Root
$''$	-	Second derivative
$\frac{d^2}{du^2}$	-	Second derivative with respect to u
$W_n(.)$	-	Set of continous and nondecreasing functions on $[0, \infty)$
$\mathbb{N}$	-	Set of natural numbers
$\mathbb{Z}^+$	-	Set of positive integers
$\mathbb{R}^+$	-	Set of positive real numbers
$C([0, D])$	-	Set of continous functions in the interval $[0, D]$
:	-	Such that
$\sum_{i=1}^{n}$	-	Summation from index $i = 1$ to $i = n$
τ	-	Time delay
$'''$	-	Third derivative
$H_m(.)$	-	The He's polynomial
$[A]^T$	-	Transpose of a matrix A

LIST OF APPENDICES

CHAPTER 1

INTRODUCTION

1.1 Introduction

Delay differential equation (DDE) is a branch of functional differential equations arising in several applications from various fields of studies, which include, biology, population dynamics, chemistry, physics, control theory, engineering, and many others [1–5]. Over the years, ordinary differential equations (ODEs) are usually used as a fundamental tool in real-life modelling problems. The formulation of models based on ODEs considered only the present state of physical problems. However, in some situations, the mathematical formulation of real-life problems needs to consider both the present and past states of the system behaviour. Therefore, the use of ODEs models in the design and analysis of such systems might lead to poor performance.

Contrary to ordinary differential equations (ODEs), the derivatives of the independent variables in delay differential equations (DDEs) at a certain time are expressed in terms of function's values at previous time. For this reason, DDEs are used in the various applications instead of ODEs. Therefore, DDEs provide an efficient mathematical tool to model various phenomena from real-life problems. Hence, researchers in various fields adopt the use of DDEs in modelling the physical process behaviour which contains the present information and some history of the system.

In recent years, many analytical methods have been used to obtain an approximate solution of DDEs [6–14]. It is observed from the literature that, most of these methods have experienced a series of challenges in finding a convergent approximate analytical solution of nonlinear DDEs more especially system of such equations. These include divergence of the result as time increases, restrictive assumptions, over dependency of small or large parameters, and lengthy with tedious calculations. Therefore, nonlinear systems of DDEs are very difficult to solve analytically [15]. Thus, obtaining the solution of any problem which mathematical models governed by these types of equations has become a challenging task. Therefore, researchers adopt the use of numerical methods as the best approach to approximate the solutions of these equations [6, 16–19].

Liao [20] introduced the Homotopy analysis method (HAM) to solve various forms of nonlinear problems. The idea of HAM comes from Homotopy, fundamentals in topology as well as differential geometry. The advantage of HAM over perturbation and some other non-perturbation techniques is that HAM does not require the existence of small or large parameter and also provides a convenient way to modify the traditional Homotopy, control and adjust the interval of convergence through the use of an auxiliary parameter for the series solution of problems. This auxiliary parameter is called a convergence controller and is one of the factors that determine the convergence of the series solutions of problems in HAM.

Consequently, HAM is applied to solve different problems. For instance, the method is used in [21] to derive a solution for the classical problem of nonlinear progressive waves in deep water. A Maclaurin series expansion provides a successive approximation of the solution through repeated application of a differential operator with the initial trial as the first term. An algorithm based HAM is developed in [8] for the solution of nonlinear ordinary differential equations of fractional order. The proposed algorithm presents the procedure of constructing the set of base functions and gives the high-order deformation equation in a simple form. The analysis is accompanied by numerical examples. The analytic approach based on HAM is also proposed in [22] to solve a nonlinear model of combined convective and radiative cooling of a spherical body. Likewise, an explicit series solution is given, which agrees well with the exact or numerical solutions. Liao [23] provides the basic concept and theorems of HAM as well as important rules associated with the method.

Khan and Khan [24] was first to define an integral transform, called N-Transform, and later, Belgacem and Silambarasan [25, 26] renamed it as Natural Transform (NT), defined its inverse and provided the details of its fundamental properties and application. The transform is derived from the renowned Fourier integral, which converges to either Laplace transform or Sumudu transform; which depends on the values of the transform variables. Therefore, NT generalized both the theory of Laplace and Sumudu transforms and is capable of handling every problem of Laplace or Sumudu. Subsequently, solutions to different types of differential and integral equations are obtained using the Natural transform. Therefore, this transform is applied to solve Maxwells equations [27] and Fluid flow problems [24] and eventually used to obtain solutions of integral equations for distribution spaces [28].

The He's polynomial was introduced by Ghorbani [29] based on the Homotopy perturbation method (HPM) to overcome difficulties of computing the nonlinear terms of various nonlinear problems. Thus, this polynomial makes the solution procedure of HPM easier, more effective, and straightforward.

In this research, a new analytical method is established based on HAM and NT. By this technique, the He's polynomial is modified to ease the computational difficulties of nonlinear terms of DDEs. The introduced method gives solutions to different classes of nonlinear DDEs in a series form, which converges to the exact solution or approximate solution using a few numbers of terms. Unlike the existing methods, the developed method provides a solution to a different form of nonlinear DDEs without linearization, perturbation, or restrictive assumptions. The technique also adjusts the interval of convergence for the series solution, avoids rounding-off of errors, and reduces the computational size compared to previous methods. The convergence analysis of the approach is also investigated to guarantee the convergence of the results produced by this newly developed method.

1.2 Research Background

Numerous similarities exist between the theories of ordinary differential equations (ODEs) and that of DDEs. In fact, some analytical methods for solving ODEs are extended to DDEs. However, due to differences that exists among the nature of these equations, such methods can only handle a linear and certain type of nonlinear DDEs. Therefore, nonlinear DDEs are very difficult to solve analytically, specifically systems of these equations, due to their special transcendental nature. Hence, they are mostly handling by numerical methods, most especially in a more complicated computational domain. This is one of the significant advantages of numerical techniques in solving nonlinear DDEs over analytical ones that often handle these equations in a simple domain. However, numerical methods also experienced some difficulties in solving nonlinear problems, for instance, if the problem contains some singularities or have multiple solutions [23]. In addition, it is very difficult to have a complete and essential understanding of nonlinear problems from numerical solutions. So, in the models of sciences and engineering, one can easily determine the important variables and how they are related to others using an analytical solution. Therefore, both numerical and analytical methods have their advantages as well as limitations. Hence, it is not vital to consider one and disregard the other.

Some perturbation methods also play a significant role in solving nonlinear DDEs based on the existence of small or large parameters, called perturbation quantities [6,10,11]. However, the existence of these quantities restricts the application of perturbation methods on nonlinear DDEs, since not all of such equations have those quantities. Also, in some cases, when nonlinearity of DDEs becomes very strong, the analytical approximation produced by perturbation methods is often broken down, and this brings another restriction of these techniques in solving nonlinear problems [23]. Consequently, perturbation approximations are only effective for nonlinear DDEs that have a weak nonlinearity.

In recent years, in order to avoid the dependence of either small or large parameters, many nonperturbation methods are introduced to solve nonlinear DDEs. Among these, the following are mentioned: Adomian decomposition method [30], variational iteration method [31], optimal Homotopy asymptotic method [7], and polynomial least square Method [12]. Others include the Taylor method approach [13, 32], Taylor collocation method [33], and implicit block method [14] are equally used to solve various types of DDEs. The approximate analytical solutions produce by most of these methods are given in the form of a polynomial series. It is well known that the power series possess a small region of convergence, and they are not always an efficient set of base functions for the approximation of nonlinear problems. Thus, it is of more interest to have acceleration methods that expand the convergence regions and use the best set of base functions for the series solutions of these equations.

In summary, none of the perturbation methods or non-perturbation methods can provide a convenient way (that is to allows the effective control of the region of convergence and rate of convergence of a series solution of a nonlinear problem) to adjust and control the convergence region and the convergence rate of the series solution. Likewise, the freedom to use a different set of base functions for efficient approximation of nonlinear DDEs has not been provided. Therefore, since the effective analytical approximation of nonlinear DDEs cannot be adequately taken into account; thus, it is required to establish some new analytic methods for such equations. An efficient analytical method, namely HAM, was introduced in [20] based on Homotopy, fundamentals in topology, and differential geometry. By this method, the existence of either small or large parameters is not needed for the series approximation of nonlinear problems. Another advantage of HAM over the previous methods is that it provides a convenient way to adjust and control the convergence region and rate of convergence of the series solution. Also, by this method, one can use a different set of base functions to approximate nonlinear problems. Therefore, HAM has been successfully used to solve a wide number of nonlinear ODEs and partial differential equations (PDEs) (see [21, 34–42]).

Recently, the concept of HAM is employed to solve different types of DDEs such as nonlinear fractional differential equations [8], neutral functional-differential equations with proportional delays [19], logistic delay differential equation [43], economics model [44], system of DDEs [45], and so on. However, most of the HAM application for nonlinear DDEs are mainly on numerical treatment; that is, the method only handle linear, single nonlinear and certain types of nonlinear systems of DDEs for analytical approximation. Furthermore, Series of challenges are observed in the application of HAM for analytical treatment of nonlinear DDEs in particular higher-order systems of these equations. Some of these include:

(i) The use of some linear operators for evaluating the linear part is very lengthy in the calculation, and this led to time-consuming in obtaining the higher-order approximation.

(ii) Lack of direct and simplified approach for evaluating different types of nonlinear terms.

(iii) Divergent of the result as time increases, since the convergence of the series solution in HAM does not only depend upon a time but also on the convergence controller.

On the other hand, NT has recently become an active topic in research due to its vast application in solving different types of differential and integral equations. NT is an integral equation that can either reduce to Laplace transform [46] or Sumudu transform [47,48]. This integral equation is a linear operator suitable in solving higher-order differential and integral equations [24]. However to the best of the author's knowledge, the application of NT to DDEs is yet to be considered by researchers.

In view of this, the present research combined these important methods of HAM and NT to come up with a robust analytical approach, with the aims to address such challenges and provide solutions to different classes of nonlinear DDEs using a small number of terms, adjust the interval of convergence of the series solution and avoid rounding-off of errors. The idea behind this combination is that the NT is used as a linear operator for evaluating the linear parts of DDEs. So, this reduce the computational size as compared to use of different types of linear operators in obtaining the higher-order approximations. Meanwhile, the concept of HAM provides the introduced method with a convenient way to adjust and control the convergence region and the rate of convergence of the series solution. There is also the freedom of using a different set of bases functions that can give an efficient analytical approximation to such equations. Finally, the He's polynomial is adjusted in order to ease the computational difficulties of nonlinear terms.

1.3 Problem Statement

Delay differential equations play an important role in describing physical phenomena from various fields of studies. These equations have drawn the interest of many researchers in modelling and explaining a different aspect of real-life problems. Many methods have been used to obtain a solution of various forms of DDEs. However, many of these methods have encountered a series of challenges in finding a convergent approximate analytical solution of nonlinear DDEs. These includes, unnecessary linearization, lengthy calculation, and over-dependency of perturbation quantities such as small or large parameters.

Consequently, the importance of HAM over perturbations and some other non-perturbations method makes it useful in finding approximate analytical solutions of nonlinear problems. The concept of HAM does not require the existence of small or large parameters and provides a convenient way to adjust and control the convergence region of the series solution of nonlinear problems. However, despite the vast application of HAM on nonlinear problems, only a few attempts were made on nonlinear DDEs due to some challenges encountered by the method in handling them. Some of these includes, divergence of the result as time increases, inappropriate selection of linear operators for evaluating linear parts, and lack of direct and simplified approach for evaluating different types of nonlinear terms. Therefore, the method can only handle linear and certain types of nonlinear DDEs.

Hence, the problem of finding a convergent approximate analytical solution of nonlinear DDEs, especially the higher-order nonlinear systems of these equations, remains an open problem in the qualitative theory of differential equations. Thus, the HAM concept requires some modifications for the analytical treatment of different classes of nonlinear DDEs. To overcome these problems, it has become necessary to develop new mathematical methods to obtain an approximate analytical solution for such equations. Therefore, in this research, the following research questions are needed to be answered.

(i) How to develop a new analytical method for solving n^{th}-order nonlinear DDEs?

(ii) Is it possible to extend the idea of the newly developed method in (i) to to n^{th}-order nonlinear system of DDEs?

(iii) How to determine the convergence rate of the newly developed methods in (i) and (ii)?

(iv) How to validate the efficiency of the newly developed method?

1.4 Research Objectives

This research aims to develop a new analytical method from the combination of the HAM and Natural transform to solve different classes of nonlinear DDEs using He's polynomials. Thus, the research has the following objectives:

(i) To develop a new analytical method for solving n^{th}-order single and system of nonlinear Retarded delay differential equations (RDDEs) and Neutral delay differential equations (NDDEs) with proportional and constant delays.

(ii) To establish the convergence analysis of the introduced method using the existing theorems of HAM and differential properties of Natural transform.

(iii) To determine the validity and applicability of the newly developed method on various problems and some existing models of real-life application resulting in different classes of nonlinear DDEs and to compare the results with those obtained by some existing methods.

1.5 Scope of the Study

This research focuses on developing a new analytical approach for solving initial value problems of different classes of nonlinear DDEs. Thus, the research considered both single and system of nonlinear DDEs of retarded and neutral types with constants and variables coefficients. However, the convergence of the series solution based on the introduced method is within a restricted subset of the domain of the Problem (the region of convergence). Likewise, the real constant time delay and proportional delay are used in this research. The Wolfram Mathematica 08 version is used for algebraic computations and drawing of graphics when needed.

1.6 Research Significance

The present work gives more attention on establishing a new mathematical method for analytical treatment of nonlinear DDEs from the combinational form of two strong methods of HAM and NT using He's polynomial. The introduced method provides a better approximate analytical solution to various types of nonlinear DDEs inform of a rapidly convergent series. Therefore, the present research help to obtain an efficient analytical approximation of many problems that lead to different classes of nonlinear DDEs. This study is also applicable to various fields of studies such as biology, chemistry, engineering, medicine, economics, and many others to analyse and solve their models described by such equations.

1.7 Research Methodology

This research commences with a review of the existing methods of solutions for DDEs by numerous researchers with the aims to highlight the strength and limitations of each method in providing the solution to various types of DDEs. However, the literature shows that the application of most of these methods for nonlinear DDEs are mainly on numerical treatment. So, the analytical approximation for most of such equations (especially higher-order nonlinear system of DDEs) cannot be efficiently obtained by using the well-known methods [19]. Therefore, the concept of HAM is of great importance here. It is obvious that, HAM is a powerful analytic method for deriving solutions to different forms of nonlinear problems [23]. The application of HAM on various types of nonlinear problems can be found in [21, 34–42]. However, despite the importance of HAM in solving different kinds of nonlinear problems but its application is less on DDEs due to some difficulties experienced by the method in finding a convergent approximate analytical solution of nonlinear DDEs. These include, an improper choice of linear operators, complexity in computing nonlinear terms, and divergence of solution for a large value of time.

It is well known that Natural transform method combines the futures of both Laplace transform and Sumudu transform [49] and therefore converges to either one of these transforms, this depends on the transform variables. Thus, NT is considered as an active transform because it handles all the problems that Laplace and Sumudu come across. This integral equation is a linear operator derived from the renowned Fourier series, and effectively solves higher-order ODEs and Partial Differential Equations (PDEs) (see [28, 49–53]). Despite the importance of delay differential equations in describing models of of physical process from a different aspect of real-life problems, but there is no attempt to apply NT to such equations. This research intends to combine HAM and NT to come up with a new analytical approach for solving different classes of nonlinear DDEs. With this method, the concept of NT is introduced to DDEs, where NT and its fundamental properties are applied to linear part of the different classes of nonlinear DDEs to obtain the simplified form of the equations. Whereas, the concept of

HAM is used to derive the generating function of the simplified form of the equations. Just like HAM, the resulting method provides a convenient way to adjust and control the convergence region and the rate of convergence of the series solution. There is also a freedom to use a different set of base functions as well as an auxiliary parameter to have a better approximation of nonlinear DDEs.

There is no direct approach for the computation of nonlinear terms in both the concept of HAM and NT. Therefore, in this research, the He's polynomial is modified to ease the computational difficulties of nonlinear terms of different classes of DDEs. The method produced solutions in the form of a rapidly convergent series with a minimal error which leads to the exact or approximate solution using a few iterations. This approach is first established based on first-order, then to second-order, and finally, generalized to n^{th}-order single nonlinear DDEs of retarded and neutral type for both single and multiple numbers of proportional and constant delays. Later, this idea is extended to n^{th}-order systems of those types of nonlinear DDEs. The convergence analysis of the developed method is established by using the existing theorems of HAM and the differential properties of NT. Finally, the proposed method's effectiveness is verified by applying the method to various problems as well as some existing models of real-life applications. The results are compared with those obtained by some previous methods. The comparison checked the complexity of the developed method, computational size, as well as the behaviour of the maximum estimated error and the length of the interval of convergence of the series solution with other methods Figure 1.1 provides the pictorial form of the introduced method.

1.8 Thesis Organization

The present research comprises of six chapters, and each chapter begins with a brief introduction and provides a conclusion at the end. Chapter 1 gives a brief background that laid down the foundation and motivation for this research. The chapter also provides the problem statement in the form of study questions and the proposed

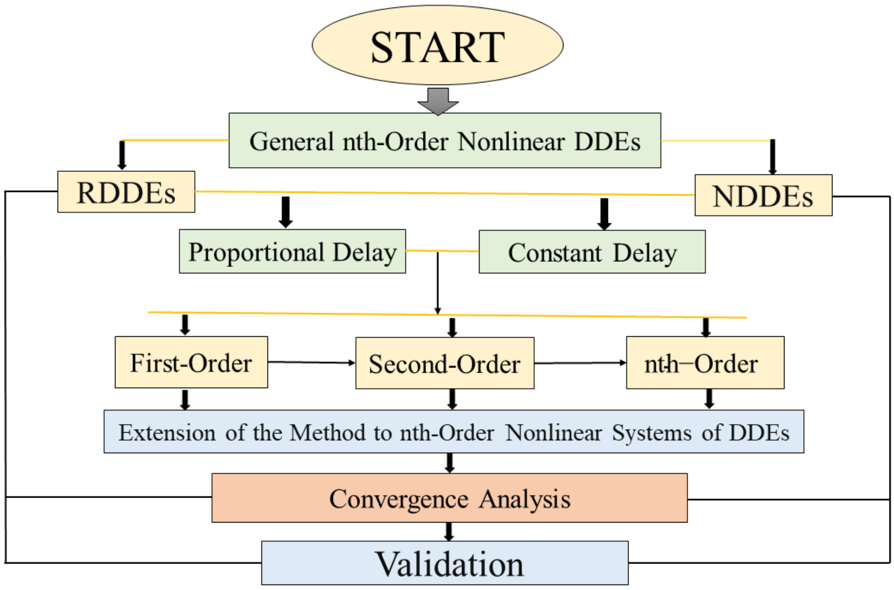

Figure 1.1 Summary of the Method Development

objectives of the research. Finally, the scope, significance, and the methodology of the research are all contained in this chapter.

Chapter 2 provides a review of various works related to DDEs, the Homotopy analysis method, and Natural transform. This chapter has seven sections, and some of which are attached with subsections. It began with the introduction and followed by a history of DDEs. The third section comprises of different classifications, definitions, and theorems associated with RDDEs and NDDEs. In Section 4, the comprehensive review of the methods of solving DDEs is given. The aim is to highlights the strengths and limitation of each method, and based on such restrictions, the research problem of the present research has been established. The ideas and basic theorems of HAM and NT are discussed in the fifth and sixth sections. The last section gives a conclusion for the whole chapter.

The main contributions of this research are laid in the next three chapters of the thesis. In Chapter 3, the algorithm of the new approach is developed for solving nonlinear retarded DDEs with both proportional and time constant delays. This result is extended to handle systems of nonlinear RDDEs, where the He's polynomial is modified to suit the computation of nonlinear terms of the systems of such equations.

Some analytical examples are given to illustrate the applicability and efficiency of the developed algorithms

In Chapter 4, the concept of Natural transform is introduced into NDDEs. The Natural transform for both proportional and constant time delays of NDDEs was successfully defined. By these definitions, the Natural transforms of NDDEs and their derivatives are derived accordingly. Based on this, the analysis of the deveploped method for NDDEs is established and later extended to nonlinear systems of the equations. To demonstrate the applicability of the approach some problems are solved analytically.

Chapter 5 gives the solution to two delay models using the presented approach. The solution of the advanced Lorenz system of RDDEs, which is the mathematical model of physical problems, was first considered. The second one is the spread of the infectious diseases model, which arises from biological sciences, and it is also a nonlinear system of RDDEs. It has turned out that the approach gives rise to simply obtainable solutions to these models. Furthermore, based on the residual error functions, it was shown that a good approximation is obtained from three numbers of iterational terms of the developed algorithm.

Chapter 6 gives the conclusion of the work and summarized the whole thesis. The summary of work highlights how the contributions in line with the objectives of the research become achievable. Lastly, the recommendations which suggest the direction for future research are also rendered in the chapter. Figure 1.2 gives the organization of the thesis schematically.

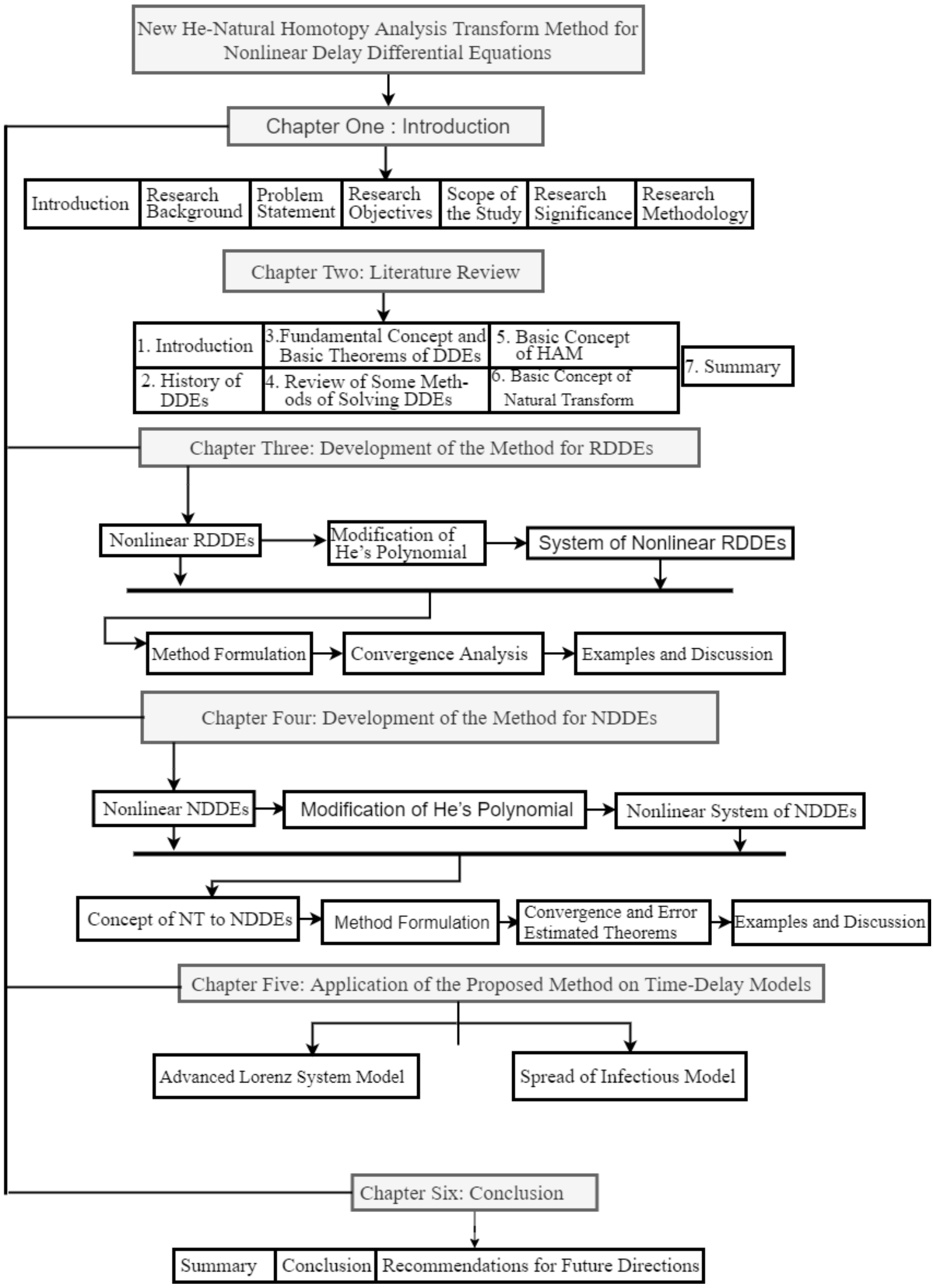

Figure 1.2 Thesis Organization

CHAPTER 2

LITERATURE REVIEW

2.1 Introduction

This chapter consists of six sections that begins with a brief history of DDEs and followed by the fundamental concept and basic theorems of delay differential equation. Some methods of solving various forms of DDEs are reviewed in the third section. The fourth section provides the basic idea and fundamental rules related to the Homotopy analysis method in solving nonlinear problems. In addition, an overview of the related works from the literature in finding solutions of DDEs using HAM is also provided. The last section reviewed some applications of NT in solving different kinds of differential and integral equations. The definitions, theorems, and some important properties of Natural transform are also given.

2.2 History of Delay Differential Equations

In 18th century Laplace and Condorcet [54] introduced the aspect of delay in Differential Equation (DE). DDEs are used as a fundamental tool when the elementary models based on ODEs have failed. Therefore, DDEs have been widely applied to model real-life problems of different areas of sciences, engineering, and many other fields. Some of these include the study of the stability of the characteristics of a feedback control system model of five differential equations with delays in both the state and control variables [1], the utility stability of the high-dimensional harmonic balance for locating limit cycles of second-order delay-differential equations as provided in [2]. The time delay model that described the tumour-immune interaction as in [4]. Others include the bacterial cell growth model [5], the delay model on

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1. **Barde, A.**, & Maan, N. (2019). Efficient Analytical Approach for Nonlinear System of Delay Differential Equations. Computer Science, 14(3), 693-712., http://ijmcs.future-in-tech.net. (Indexed by SCOPUS)
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1. **Barde, A.**, Maan, N. (2019). A Natural Homotopy Analysis Method for Nonlinear Delay Differential Equations. In 2019 3rd Asia International Multidisciplinary Conference (AIMC2019) (pp. 86-90). Science Proceedings Series. https://doi.org/10.31580/sps.v1i2.680. (Indexed by SPS)
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