MULTISCALE HYBRID FINITE ELEMENT AND FINITE VOLUME METHOD FOR HIGH GRADIENT BOUNDARY VALUE PROBLEMS

OLAIJU OLUSEGUN ADEYEMI

UNIVERSITI TEKNOLOGI MALAYSIA

MULTISCALE HYBRID FINITE ELEMENT AND FINITE VOLUME METHOD FOR HIGH GRADIENT BOUNDARY VALUE PROBLEMS

OLAIJU OLUSEGUN ADEYEMI

A thesis submitted in partial fulfilment of the
requirements for the award of the degree of
Doctor of Philosophy

Faculty of Science
Universiti Teknologi Malaysia

DECEMBER 2021

DEDICATION

This thesis is dedicated to my lovely daughter,

OLAIJU ABIMBOLA LOIS.

ACKNOWLEDGEMENT

I praise and thank God, the Almighty, for His blessings and for helping me throughout the four years of my course of study. I thank Him for bringing me to a successful end of the program and for helping me complete my research project.

I want to express my deep and sincere gratitude to my research supervisor, Associate Professor Dr Yeak Su Hoe, for allowing me to do the research and for providing invaluable guidance throughout the research. His dynamism, vision, sincerity, and motivation have deeply inspired me. He has taught me the methodology to carry out the research and present the research work clearly. It was a great privilege and honour to work and study under his guidance. I am incredibly grateful for what he has offered me. I would also like to thank him for his friendship, empathy, and great sense of humour.

I am also indebted to The Tertiary Education Trust Fund (TETFund) for funding my PhD study and the management of the Federal Polytechnic Ilaro for allowing time for the study. I thank the Rector of the FPI, Dr Aluko Olusegun and Mr Durowaye Adekunle for their understanding.

It is my privilege to thank my wife, Mrs Olaiju Seun Tayo and my sons and daughter, Olaiju Ayomikun, Olaiju Darasimi and Olaiju Abimbola Lois, for their constant encouragement throughout my research period. Special appreciation to my indefatigable sororal nieces and nephew and their parents, Adenijo Aderonke, Adenijo Bambi, Adenijo Adeyemi, Mrs Adenijo Oluwatoyin and Mr Adenijo Sunday.

I am incredibly grateful to my family for their love, prayers, care, and sacrifices. Also, I express my thanks to my sisters and brothers and my mother-in-law for their support and valuable prayers. My sincere appreciation also extends to my colleagues, Dr Olapeju Olasunkanmi, Dr Fabi Jonathan, Dr Amusuk Damboi, Dr Akeremale collins, Dr Ogunbode Michel and others who have assisted on various occasions.

I am also grateful to late Fagoyinbo Idowu Sunday, Mrs Fagoyinbo Abosede, Mr Ezekiel Donaldson, Dr Akinde Mukail, Dr Akinleke Wasiu, Mr Aladelusi Kehinde, Dr Oluseye Abiodun, Engineer Aikulola Olumuyiwa, Mr Salako Olufemi and the entire staff of the Mathematics and Statistics Department of the Federal Polytechnic, Ilaro for their help and encouragement when the going was tough. May Almighty God be with you all and support you, especially in your moments of need.

ABSTRACT

A multiscale hybrid finite element and finite volume method (MSHFEFVM) was introduced for high gradient boundary value problems by coupling an adaptive finite element and node centred finite volume schemes. Starting with the traditional four-node finite element method, additional nodes were inserted automatically at high gradient regions by an adaptive algorithm based on refinement criteria. A posteriori error estimation and error indicator were formulated. The error estimation was residual-based, while the error indicator was gradient-based. Using the information from the gradient-based error indicator, a p-refinement indicator was used to decide whether a given element should be refined or not via adaptive algorithm. Two sets of elements were used to design the adaptive algorithm which are the regular elements and transition elements. The regular elements are the linear and quadratic elements, while the transition elements are the elements having both quadratic and linear sides. These elements are useful in transitioning from linear to quadratic elements during the implementation of the adaptive algorithm. The coupling resulted in a multiscale finite element method (MSFEM). The MSFEM was applied to some two-dimensional high gradient problems with promising results. The MSFEM was extended to solve the time dependent partial differential problems. The results obtained showed good agreement with the analytical results. A node centred finite volume method was coupled with the MSFEM to form a MSHFEFVM based on concurrent continuum-continuum coupling using a handshake coupling technique that allows information passing between the two coupled methods on a fly. The proposed hybrid technique was first applied to some two-dimensional localised high gradient problems with available analytical solutions. This application was necessary to analyse and validate the performance and accuracy of the MSHFEFVM. The obtained numerical results from the analysis in terms of error and execution time showed an encouraging performance of the scheme compared to the traditional finite element, the node centred finite volume and the MSFEM. Finally, the MSHFEFVM was applied to two standard localised high gradient problems and two engineering problems, which are electrostatics and torsion problems. The application showed a promising performance of the new scheme. The numerical results show that the combination of these two techniques can help to solve high gradient problems with accuracy and minimum execution time.

ABSTRAK

Unsur terhingga hibrid multiskala dan kaedah isipadu terhingga (MSHFEFVM) diperkenalkan untuk menyelesaikan masalah nilai sempadan berkecerunan tinggi dengan menggandingkan unsur terhingga secara adaptif dan skema isipadu terhingga berpusat pada nod. Bermula dengan kaedah unsur terhingga empat nod tradisional, nod tambahan dimasukkan secara automatik di kawasan kecerunan tinggi oleh algoritma adaptif berdasarkan kriteria penyempurnaan. Anggaran ralat posteriori dan penunjuk ralat dirumuskan. Anggaran ralat adalah berdasarkan sisa, sementara penunjuk ralat berdasarkan kecerunan. Menggunakan maklumat dari penunjuk ralat berdasarkan kecerunan, penunjuk penambahbaikan p digunakan untuk memutuskan sama ada elemen tertentu harus diperhaluskan atau tidak melalui algoritma adaptif. Dua set unsur digunakan untuk merancang algoritma adaptif iaitu unsur biasa dan unsur peralihan. Unsur biasa adalah unsur linear dan kuadratik, sementara unsur peralihan adalah unsur yang mempunyai sisi kuadratik dan linear. Semua unsur ini berguna dalam peralihan dari unsur linear ke kuadratik semasa pelaksanaan algoritma adaptif. Gandingan ini menghasilkan kaedah unsur terhingga multiskala (MSFEM). MSFEM diterapkan pada beberapa masalah kecerunan tinggi dua dimensi dengan keputusan yang memuaskan. MSFEM diperluas untuk menyelesaikan masalah pembezaan separa yang bersandar pada masa. Keputusan yang diperoleh menunjukkan persetujuan yang baik dengan hasil analisis. Kaedah isipadu terhingga berpusat pada nod digandingkan dengan MSFEM untuk membentuk MSHFEFVM berdasarkan gandingan kontinum-kontinum serentak menggunakan teknik gandingan jabat tangan yang membolehkan maklumat disampaikan antara kedua-dua kaedah terganding dengan cepat. Teknik hibrid yang dicadangkan pertama kali diterapkan pada beberapa masalah berkecerunan tinggi dua dimensi yang disetempatkan dengan penyelesaian analisis yang tersedia. Aplikasi ini diperlukan untuk menganalisis dan mengesahkan prestasi dan ketepatan MSHFEFVM. Keputusan berangka yang diperoleh dari analisis dari segi ralat dan masa pelaksanaan menunjukkan prestasi skema yang memberangsangkan berbanding dengan unsur terhingga tradisional, isipadu terhingga berpusat dan MSFEM. Akhirnya, MSHFEFVM diaplikasikan pada dua masalah kecerunan tinggi yang disetempatkan dan dua masalah kejuruteraan, iaitu masalah elektrostatik dan kilasan. Pengaplikasian ini telah menunjukkan prestasi skema baharu yang memberangsangkan. Keputusan berangka menunjukkan bahawa gabungan kedua-dua teknik tersebut dapat membantu menyelesaikan masalah kecerunan tinggi dengan tepat dan masa pelaksanaan minimum.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

LIST OF ABBREVIATIONS

FEM	-	Finite Element Method.
FVM	-	Finite Volume Method.
MSFEM	-	Multiscale Finite Element Method.
MsFVM	-	Multiscale Finite Volume Method.
MSHFEFVM	-	Multiscale hybrid Finite Element finite Volume Method.
FEA	-	Finite Element Analysis.
PDE	-	Partial Differential Equation.
XFEM	-	Extended Finite Element Method.
GFEM	-	Generalized Finite Element Method.
Hp-FEM	-	Higher Polynomial Refinement Finite Element Method.
DG-FEM	-	Discontinuous Galerkin Finite Element Method.
p-AFEM	-	Polynomial Adaptive Finite Element Method.
te	-	Transition element.
TH	-	Threshold.
CP	-	Corner Point.
e	-	Element.
CM	-	Connection Matrix.
Sol	-	Solver.
LtoG	-	Local to Global.
Flops	-	Floating-point operation per second.
LFEM	-	Linear FEM.
QFEM	-	Quadratic FEM.
RK4	-	Runge-Kutta of order 4.

CR	-	Critical Region.
EB	-	East Boundary.
SB	-	South Boundary.
WB	-	West Boundary.
NB	-	North Boundary.
AEB	-	Adjusted East Boundary.
RK4	-	Runge-Kutta of order 4.
FEV	-	Coupled Finite Element and Finite Volume.
AWB	-	Adjusted West Boundary.
LFE	-	Length of FE.
LFV	-	Length of FV.
NIST	-	National Institute of Standards and Technology.

LIST OF SYMBOLS

u	-	Displacement.
Ω	-	Domain of Interest.
Ω_{FE}	-	The domain of Finite Element Method.
Ω_{FV}	-	The domain of Finite Volume Method.
K	-	Stiffness Matrix.
M	-	Mass Matrix.
F	-	Force vector.
F^{FEV}	-	FEV Force vector.
K_{FE}	-	FEM expanded Stiffness Matrix.
K_{FV}	-	FVM expanded stiffness Matrix.
K^{AEB}	-	FEM stiffness Matrix with the adjusted east boundary.
K^{AWB}	-	FVM stiffness Matrix with the adjusted west boundary.
K_{FEV}	-	Finite Element Volume Stiffness Matrix.
$\kappa(A)$	-	Condition number of Matrix A.
$\|e\|_2^{LFE}$	-	Linear FEM Norm2error.
$\|e\|_2^{QFE}$	-	Quadratic FEM Norm2error.
$\|e\|_2^{MS}$	-	Multiscale FEM Norm2error.
$\|e\|_2^{FV}$	-	Multiscale hybrid FEM Norm2error.
e_{FE}^{L}	-	Linear finite element residual error.
e_{FE}^{Q}	-	Quadratic finite element residual error.
R^{L}	-	Linear FEM residual.
R^{Q}	-	Quadratic FEM residual.
Γ	-	FEM boundary.
Γ_N	-	Neumann boundary.
Γ_D	-	Dirichlet boundary.

τ	-	Diffusion coefficient.
N_i	-	ith shape function.
N_i^L	-	Linear ith shape function.
N_i^Q		Quadratic ith shape function.
nx	-	Number of Element in the x-direction.
ny	-	Number of Element in the y-direction.
g_n	-	The gradient at node n.
nel	-	The number of elements.
nno	-	Number of Nodes.

LIST OF APPENDICES

CHAPTER 1

INTRODUCTION

1.1 Motivation

With the rapid advance in computer power, numerical simulations have become a very important method for solving questions in engineering and science. Instead of adopting the traditional theoretical practice using assumptions and approximations to simplify problems, this numerical approach addresses the original problem in all its detail with little assumptions. Providing validation for theories, providing insight into experimental results, and providing help in interpreting or even discovering new phenomena on varied scales, hence, it is a valuable instrument in science (Jebahi *et al*., 2016; Zeng and Qin, 2018).

A numerical simulation can typically be divided into four scales. The nanoscopic scale, where phenomena related to the behavior of electrons become significant, the microscopic scale, where phenomena related to the behavior of atoms are considered. The mesoscopic scale at which defects in lattices are studied. The macroscopic scale which allows continuum mechanics to explain macroscopic phenomena. Each scale has been addressed by several numerical methods (Weinan and Lu, 2011; Jebahi *et al*., 2016; Zeng and Qin, 2018).

There are two classes of numerical methods: discrete methods (DM) and continuum methods (CM). Discrete mechanics is the first class and covers the first three scales. They are quantum mechanical methods (QMs) used for nanoscopic analysis, atomistic methods (AMs) used for microscopic analysis, and mesoscopic discrete methods (MDMs) used for mesoscopic analysis. Although the class can provide extremely accurate results, but it is considerably time-consuming and is only suitable for small physical systems (Weinan and Lu, 2011; Jebahi *et al*., 2016; Zeng and Qin, 2018). Continuous methods (CMs) are based on continuum mechanics and are focused on solving macroscopic problems. Nonetheless, handling additional episodes of fine scale phenomena is generally necessary. Although continuous methods are less accurate than discrete ones, but they are relatively inexpensive and flexible for large systems (Jebahi *et al*., 2016; Zeng and Qin, 2018).

Simulations of modern materials have been challenged by multiscale phenomena. Such phenomena require on one scale an extremely accurate and computationally expensive description, and on another scale a coarser description that avoids prohibitively large calculations. Since all DMs and CMs methods alone are not sufficient to describe the entire system, so coupling approaches combining different methods at different scales would be beneficial in distributing computation effort as needed. As a result, multiple multiscale coupling approaches have since been developed to assess mechanical behavior of materials at all relevant scales while retaining accuracy of the individual approaches at each scale (Jebahi *et al*., 2016; Zeng and Qin, 2018). From all developed multiscale approaches, the concurrent continnum-continnm multiscale methods are very few. Most of the concurrent models in the literature are coupling of atomicity and countinnum models (Fish, 2010; Yamashita *et al*.,2016). Thus, the motivation for the adoption of the concurrent continnum-continnm multiscale model in this work.

1.2 Background of the Problem

The academic interest in the adoption of multiscale methods for solving physics and engineering problems has been on the increase. This ramped interest had been inspired by factors such as the complex and 'multiscale' nature of many application problems across wide spheres of knowledge, the breakthroughs, and rapid appreciations in computing science, and essentially the imperative accuracy and efficiency that highly detailed multiscale and multi-physics problems require. It is usually too exorbitant to compute on the tiniest scale problems that integrate a wide range of scales in the coefficients or solutions. This is without regard to when sophisticated super-computers are deployed. This is largely due to the many unknowns inherent in finest scales, especially if what is of interest are relatively longer lengths and time scales.

Computing on a coarser scale, however, exclusively, may tend towards inaccuracy, as finest scale properties usually seriously impinge on coarse-scale behaviours. Moreover, the case had always been that the coarse-scale properties and guiding equations lack proper definition. It is worthy of note that existing multiscale methods are a hybrid of fine-scale and coarse-scale computational techniques aimed at efficiently resolving the most important fine-scale data without the need for a recourse to direct computation usually associated with global fine-scale problems.

As encapsulated in Car and Parrinello (1985), Zhang *et al*. (1999), and Li and Weinan (2005), the foregoing has application in solid mechanics where the coarse-scale (macroscopic) continuum theory conflates molecular dynamics on the fine-scale (microscopic) to achieve accuracy in the appreciation of macroscopic properties of materials. It also has application in computational fluid dynamics where hydrodynamics and kinetic models, which are respectively on coarse-scale and fine-scale, are integrated to capture the allocation of shocks (Le Tallec and Mallinger, 1997; Schwartzentruber and Boyd, 2006). Multiscale methods equally have application in the critical study of turbulent flow (Pal and Ganesan, 2015) and nano materials (Liu *et al*., 2004). Flow in porous media is another important application area for multiscale methods.

1.3 Statement of the Problem

Over the years, there have been satisfactory results associated with the adoption of the adaptive finite element method (AFEM) and finite volume method (FVM) is offering a panacea to various kinds of problems that exhibit rapid changes or sharp fronts or wiggles in their numerical solutions (high gradient problems). However, hybridizing the AFEM and finite volume (AFEM/FVM) methods allows the creation of multiscale models that afford the employment of different material models at different levels in different subdomains of the same system. While some parts allow for the adoption of the FV model, AFEM, on the premise of the continuum mechanics model, can be used in other parts. The FVM is ultimately suitable for modelling materials with the tendency for discontinuities and failure, mostly typified by fracture, shocks, inter alia (Fallah *et al.*, 2000). The AFEM, on the other hand, is usually involved linear and nonlinear continuous material behaviour in solving high gradient problems.

However, this work aims at treating FVM and AFEM as complementary methods in a bid to optimize the advantages of each method. This will have utility in high gradient problems, where numerical errors are inevitable with sudden modifications in numerical solutions. The study is desired to obtain and optimize the hybridized properties of AFEM and FVM. Also, to refine high gradient zones locally to enhance potential solutions' accuracy, thereby developing a numerical model that requires less computational cost and time. The research will ultimately clarify the following questions:

1. How can the advantages of both AFEM and FVM be optimized?
2. How can multiscale hybridization of AFEM and FVM be achieved?
3. How can the solution accuracy in the newly proposed multiscale hybrid method be improved?
4. Where and how can the newly proposed multiscale hybrid technique be applied?
5. How can the speed of the new multiscale hybrid technique be optimized?

1.4 The Study Objectives

The ultimate objective is to conflate the adaptive finite element method with the finite volume method to afford enhanced solutions to high gradient problems. The following are the main objectives to achieve the goal:

1. To develop an adaptive finite element method (AFEM).

2. To develop a coupled AFEM and finite volume method (FVM).

3. To compare the numerical results of the models with existing analytical solutions.

1.5 Scope of the Study

Several collections of hybrid techniques exist in literature; however, this work focuses on the coupling of *p*-adaptive finite element (*p*-AFEM) method and finite volume method to produce a multiscale hybrid technique with improved accuracy and efficiency. The formulation and application of the anticipated new technique are restricted to regular high gradient problems in two-dimension. The tools used to compute the numerical results are codes written in OCTAVE and MAPLE programming languages.

1.6 Significance of the Study

In most of the previous studies, the sequential continuum-continuum multiscale models were considered chiefly. This has created a wide gap in the study of the concurrent continuum-continuum models. In order to bridge the gap, this work is based on coupling a concurrent continuum-continuum multiscale model. The coupling of two continuum methods, the Lagrange method (AFEM) and the Eulerian method (FVM), is considered. The work is premised on the accuracy of AFEM and the speed of FVM to develop an improved mathematical model and algorithm. The proposed hybrid technique, envisaged as the research outcome, can produce higher precision results, and require less time for numerical operations. This makes it a veritable technique for solving large domain size and deformations problems. The acquired results equally have practical applications in studying high gradient problems in both engineering and mathematical fields. In addition, the outcomes can stimulate new gaps that can be leveraged upon for further research in related areas.

1.7 The Thesis Layout

The work consist of six chapters and the chapters are structured as follows: The background of the problem is introduced in chapter 1, followed by the problem statement and the study objectives. Also, analysis is done on the study scope and significance. Finally, the layout of the thesis is established.

Chapter 2 provides a comprehensive review of earlier studies on FE technique, FV technique, multiscale techniques, multiscale FE techniques, multiscale FV techniques, hybrid multiscale FE/FV technique, adaptive techniques, Runge-Kutta methods, high gradient boundary value problems, the intricate reasons for coupling and the research gap.

In Chapter 3, the finite element method and the finite volume methods are designed. The validation techniques and robust residual error analysis theorem based on matrix condition numbers are discussed.

Chapter 4 is based on the development of the p-adaptive finite element method (p-AFEM), time-dependent FEM, p-adaptive finite element method error analysis and the numerical validation of the techniques.

In Chapter 5, a proposed numerical technique hinged on the coupling of p-AFEM and finite volume method is designed. The formulation of error analysis for the new technique is included. The efficiency and feasibility of the proposed new technique are validated with high gradient problems having accessible analytical solutions. The new method is finally applied to two standard high gradient problems and two engineering problems namely the electrostatic and torsion problems.

Lastly, in Chapter 6, the conclusions are drawn, and recommendations for future research are illustrated.

REFERENCES

Aarnes, J. E., Krogstad, S., & Lie, K. A. (2006). A hierarchical multiscale method for two-phase flow based upon mixed finite elements and nonuniform coarse grids. *Multiscale Modeling & Simulation*, 5(2), 337-363.

Abbas, S., Alizada, A., & Fries, T. P. (2010). The XFEM for high-gradient solutions in convection-dominated problems. International journal for numerical methods in engineering, 82(8), 1044-1072.

Abbaszadeh, M., & Dehghan, M. (2020). Direct meshless local Petrov-Galerkin method to investigate anisotropic potential and plane elastostatic equations of anisotropic functionally graded materials problems. *Engineering Analysis with Boundary Elements*, *118*, 188-201.

Aboubacar, M., & Webster, M. F. (2003). Development of an optimal hybrid finite volume/element method for viscoelastic flows. *International journal for numerical methods in fluids*, 41(11), 1147-1172.

Abouelatta, M. A., Omar, A. M., & Ward, S. (2020). Optimal grid size for precipitators using finite difference method based on full multi-grid method. *Electric Power Systems Research*, 189, 106575.

Akbari, M., Asadi, P., & Behnagh, R. A. (2021). Modeling of material flow in dissimilar friction stir lap welding of aluminum and brass using coupled Eulerian and Lagrangian method. *The International Journal of Advanced Manufacturing Technology*, *113*(3), 721-734.

Akin J.E. (2008). Torsion of Solid Shafts and the Thermal Analogy. Retrieved October 9, 2021, from https://www.clear.rice.edu/mech400/TorsionSolidBars_ThermalAnalogy.pdf

Alder, B. J., & Wainwright, T. E. (1959). Studies in molecular dynamics. I. General method. *The Journal of Chemical Physics*, *31*(2), 459-466.

Alghosoun, A., Osman, A., & Seaid, M. (2018). A Finite Element/Finite Volume Method for Dam-Break Flows over Deformable Beds. *International Journal of Bioengineering and Life Sciences*, *12*(10), 395-399.

Aliabadi, S., Akbar, M., & Patel, R. (2010). Hybrid finite element/volume method for shallow water equations. *International journal for numerical methods in engineering, 83*(13), 1719-1738.

Aliabadi, S., Bigler, C., Yilmaz, E., Palle, S., & Soni, B. (2011). Hybrid finite element/volume method for 3D incompressible flows with heat transfer. *International Journal of Computational Fluid Dynamics*, 25(4), 175-189.

Alimirzaei, S., Mohammadimehr, M., & Tounsi, A. (2019). Nonlinear analysis of viscoelastic micro-composite beam with geometrical imperfection using FEM: MSGT electro-magneto-elastic bending, buckling and vibration solutions. *Struct Eng Mech, 71*(5), 485-502.

Ancilotto, F., Barranco, M., Guilleumas, M., & Pi, M. (2018). Self-bound ultradilute Bose mixtures within local density approximation. *Physical Review A, 98*(5), 053623.

André, D., Jebahi, M., Iordanoff, I., Charles, J. L., & Néauport, J. (2013). Using the discrete element method to simulate brittle fracture in the indentation of a silica glass with a blunt indenter. *Computer Methods in Applied Mechanics and Engineering*, *265*, 136-147.

Ansari, M. A., Samanta, A., Behnagh, R. A., & Ding, H. (2019). An efficient coupled Eulerian-Lagrangian finite element model for friction stir processing. *The International Journal of Advanced Manufacturing Technology*, *101*(5), 1495-1508.

Archambeault, B., & Connor, S. (2008, August). Proper model validation is important for all EMI/EMC applications. In 2008 *IEEE International Symposium on Electromagnetic Compatibility (pp. 1-8). IEEE*.

Armentano, M. G., Padra, C., Rodríguez, R., & Scheble, M. (2011). An hp finite element adaptive scheme to solve the Laplace model for fluid–solid vibrations. *Computer methods in applied mechanics and engineering*, 200(1-4), 178-188.

Ashgriz, N., & Mostaghimi, J. (2002). An introduction to computational fluid dynamics. *Fluid flow handbook, 1*, 1-49.

Atluri, S. N., & Zhu, T. (1998). A new meshless local Petrov-Galerkin (MLPG) approach in computational mechanics. *Computational mechanics*, *22*(2), 117-127.

Babuška, I., & Banerjee, U. (2012). Stable generalized finite element method (SGFEM). *Computer methods in applied mechanics and engineering*, 201, 91-111.

Babuška, I., & Dorr, M. R. (1981). Error estimates for the combined h and p versions of the finite element method. *Numerische Mathematik*, *37*(2), 257-277.

Babuška, I., & Ohnimus, S. (2001). A posteriori error estimation for the semidiscrete finite element method of parabolic differential equations. *Computer Methods in Applied Mechanics and Engineering*, 190(35-36), 4691-4712.

Babuška, I., & Suri, M. (1987). The h-p version of the finite element method with quasiuniform meshes. *ESAIM: Mathematical Modelling and Numerical Analysis-Modélisation Mathématique et Analyse Numérique*, *21*(2), 199-238.

Babuška, I., Banerjee, U., & Osborn, J. E. (2004). Generalized finite element methods—main ideas, results and perspective. *International Journal of Computational Methods*, 1(01), 67-103.

Babuška, I., Ihlenburg, F., Strouboulis, T., & Gangaraj, S. K. (1997). A posteriori error estimation for finite element solutions of Helmholtz'equation. Part I: The quality of local indicators and estimators. *International Journal for Numerical Methods in Engineering*, 40(18), 3443-3462.

Babuška, I., Ihlenburg, F., Strouboulis, T., & Gangaraj, S. K. (1997). A posteriori error estimation for finite element solutions of Helmholtz'equation—Part II: estimation of the pollution error. *International Journal for Numerical Methods in Engineering*, 40(21), 3883-3900.

Babuska, I., Szabo, B. A., & Katz, I. N. (1981). The p-version of the finite element method. *SIAM journal on numerical analysis*, *18*(3), 515-545.

Bakroon, M., Daryaei, R., Aubram, D., & Rackwitz, F. (2018). Multi-material arbitrary Lagrangian-Eulerian and coupled Eulerian-Lagrangian methods for large deformation geotechnical problems. In *Numerical Methods in Geotechnical Engineering IX* (pp. 673-681). CRC Press.

Baliga, B. R., & Patankar, S. V. (1983). A control volume finite-element method for two-dimensional fluid flow and heat transfer. *Numerical Heat Transfer*, 6(3), 245-261.

Baliga, B. R., Pham, T. T., & Patankar, S. V. (1983). Solution of some two-dimensional incompressible fluid flow and heat transfer problems, using a control volume finite-element method. *Numerical Heat Transfer*, 6(3), 263-282.

Bangerth, W., & Kayser-Herold, O. (2009). Data structures and requirements for hp finite element software. *ACM Transactions on Mathematical Software (TOMS)*, 36(1), 1-31.

Barlow, A. J., Maire, P. H., Rider, W. J., Rieben, R. N., & Shashkov, M. J. (2016). Arbitrary Lagrangian–Eulerian methods for modeling high-speed compressible multimaterial flows. *Journal of Computational Physics*, *322*, 603-665.

Bazrafkan, S., Matthai, S. K., & Mindel, J. E. (2014, September). The finite-element-centered finite-volume discretization method (FECFVM) for multiphase transport in porous media with sharp material discontinuities. In *ECMOR XIV-14th European conference on the mathematics of oil recovery* (Vol. 2014, No. 1, pp. 1-22). European Association of Geoscientists & Engineers.

Beilina, L., Karchevskii, E., & Karchevskii, M. (2017). *Numerical linear algebra: Theory and applications*. New York: Springer International Publishing.

Belytschko, T., Lu, Y. Y., & Gu, L. (1994). Element-free Galerkin methods. *International journal for numerical methods in engineering*, *37*(2), 229-256.

Berger, M. J., & Colella, P. (1989). Local adaptive mesh refinement for shock hydrodynamics. *Journal of computational Physics*, 82(1), 64-84.

Bermúdez, A., Ferrín, J. L., Saavedra, L., & Vázquez-Cendón, M. E. (2014). A projection hybrid finite volume/element method for low-Mach number flows. *Journal of Computational Physics*, 271, 360-378.

Bochev, P., Peterson, K., & Perego, M. (2015). A multiscale control volume finite element method for advection–diffusion equations. *International Journal for Numerical Methods in Fluids*, *77*(11), 641-667.

Brandt, A. (1977). Multi-level adaptive solutions to boundary-value problems. *Mathematics of computation*, 31(138), 333-390.

Brunetti, G., Šimůnek, J., & Bautista, E. (2018). A hybrid finite volume-finite element model for the numerical analysis of furrow irrigation and fertigation. *Computers and electronics in agriculture*, 150, 312-327.

Burgarelli, D., Kischinhevsky, M., & Biezuner, R. J. (2006). A new adaptive mesh refinement strategy for numerically solving evolutionary PDE's. *Journal of Computational and Applied Mathematics*, 196(1), 115-131.

Butcher, J. C. (1976). On the implementation of implicit Runge-Kutta methods. *BIT Numerical* Mathematics, 16(3), 237-240.

Calgaro, C., Creusé, E., & Goudon, T. (2008). A hybrid finite volume–finite element method for variable density incompressible flows. *Journal of Computational Physics*, 227(9), 4671-4696.

Calzado, G. R. (2015). *Truncation error estimation in the Discontinuous Galerkin Spectral Element Method* (Doctoral dissertation, Universidad Politécnica de Madrid).

Car, R. and Parrinello, M. (1985). Unified approach for molecular dynamics and density-functional theory. *Physical review letters*. 55(22), 2471.

Cardiff, P., & Demirdžić, I. (2021). Thirty years of the finite volume method for solid mechanics. *Archives of Computational Methods in Engineering*, 1-60.

Carstensen, C., Eigel, M., & Gedicke, J. (2011). Computational competition of symmetric mixed FEM in linear elasticity. *Computer methods in applied mechanics and engineering*, 200(41-44), 2903-2915.

Castelletto, N., Hajibeygi, H., & Tchelepi, H. A. (2016, August). Hybrid multiscale formulation for coupled flow and geomechanics. In *ECMOR XV-15th European Conference on the Mathematics of Oil Recovery* (pp. cp-494). European Association of Geoscientists & Engineers.

Chauhan, P., Jain, R., Pal, S. K., & Singh, S. B. (2018). Modeling of defects in friction stir welding using coupled Eulerian and Lagrangian method. *Journal of Manufacturing Processes*, *34*, 158-166.

Chen, L., & De Borst, R. (2018). Adaptive refinement of hierarchical T-splines. *Computer Methods in Applied Mechanics and Engineering*, 337, 220-245.

Chen, X., Zhang, L., Chen, L., Li, X., & Liu, D. (2019). Slope stability analysis based on the Coupled Eulerian-Lagrangian finite element method. *Bulletin of Engineering Geology and the Environment*, *78*(6), 4451-4463.

Chen, Z., & Hou, T. (2003). A mixed multiscale finite element method for elliptic problems with oscillating coefficients. *Mathematics of Computation*, 72(242), 541-576.

Chernyshenko, A., Olshahskii, M., & Vassilevski, Y. (2017, June). A Hybrid Finite Volume—Finite Element Method for Modeling Flows in Fractured Media. In *International Conference on Finite Volumes for Complex Applications* (pp. 527-535). Springer, Cham.

Chu, C. C., Graham, I., & Hou, T. Y. (2010). A new multiscale finite element method for high-contrast elliptic interface problems. *Mathematics of Computation*, 79(272), 1915-1955.

Ciarlet Jr, P., & Zou, J. (1997). Finite element convergence for the Darwin model to Maxwell's equations. *ESAIM: Mathematical Modelling and Numerical Analysis-Modélisation Mathématique et Analyse Numérique*, 31(2), 213-249.

Coelho, P. J. (2004). A hybrid finite volume/finite element discretization method for the solution of the radiative heat transfer equation. *In RADIATIVE TRANSFER-IV. Fourth International Symposium on Radiative Transfer*. Begel House Inc.

Coetzee, C. J. (2017). Calibration of the discrete element method. *Powder Technology*, *310*, 104-142.

Courant, R. (1943). Variational methods for the solution of problems of equilibrium and vibrations. *Bulletin of the American mathematical Society*. 49(1),1-23.

Cundall, P. A., & Strack, O. D. (1979). A discrete numerical model for granular assemblies. *geotechnique*, *29*(1), 47-65.

Cüneyt, S. (2012). Finite element analysis in thermofluids. *Dept. Mech. Eng., Middle East Tech. Univ., Ankara, Turkey, Tech. Rep. ME582*. Retrieved January 22, 2020. http://users.metu.edu.tr/csert/teaching_notes.htm

Daubechies, I. (1992). Ten lectures on wavelets. Retrieved January 22, 2020, from https://epubs.siam.org/doi/pdf/10.1137/1.9781611970104.fm.

Dolean, V., Jolivet, P., & Nataf, F. (2015). An introduction to domain decomposition methods: algorithms, theory, and parallel implementation. Retrieved January 22, 2020, from https://hal.archives-ouvertes.fr/cel-01100932v4/document

Dostert, P., Efendiev, Y., & Hou, T. Y. (2008). Multiscale finite element methods for stochastic porous media flow equations and application to uncertainty quantification. *Computer Methods in Applied Mechanics and Engineering*, 197(43-44), 3445-3455.

Ducobu, F., Riviere-Lorphevre, E., & Filippi, E. (2016). Application of the Coupled Eulerian-Lagrangian (CEL) method to the modeling of orthogonal cutting. *European Journal of Mechanics-A/Solids*, *59*, 58-66.

Duffy, A. P., Martin, A. J., Orlandi, A., Antonini, G., Benson, T. M., & Woolfson, M. S. (2006). Feature selective validation (FSV) for validation of computational electromagnetics (CEM). part I-the FSV method. *IEEE transactions on electromagnetic compatibility*, *48*(3), 449-459.

Duffy, A., Orlandi, A., & Zhang, G. (2018). Review of the Feature Selective Validation Method (FSV). Part I-Theory. IEEE Transactions on Electromagnetic Compatibility, 60, 814-821.

Efendiev, Y. R., Hou, T. Y., & Wu, X. H. (2000). Convergence of a nonconforming multiscale finite element method. *SIAM Journal on Numerical Analysis*, 37(3), 888-910.

Efendiev, Y., & Hou, T. Y. (2009). *Multiscale finite element methods: theory and applications* (Vol. 4). Springer Science & Business Media.

Efendiev, Y., Galvis, J., & Hou, T. Y. (2013). Generalized multiscale finite element methods (GMsFEM). *Journal of Computational Physics*, 251, 116-135.

El-Amin, M. F., Kou, J., & Sun, S. (2018). Mixed finite element simulation with stability analysis for gas transport in low-permeability reservoirs. *Energies*, 11(1), 208.

Elruby, A. Y., Nakhla, S., & Hussein, A. (2018). Automating XFEM Modeling Process for Optimal Failure Predictions. *Mathematical Problems in Engineering*, 2018.

Eymard, R., Gallouët, T. and Ele Herbin, R. (2003). Finite Volume Methods. *P.G. Ciarlet, J.L. Lions (eds)*.

Eymard, R., Hilhorst, D., & Vohralík, M. (2006). A combined finite volume–nonconforming/mixed-hybrid finite element scheme for degenerate parabolic problems. *Numerische Mathematik*, *105*(1), 73-131.

Ezeh, J. C., & Enem, J. I. (2012). Comparative study on use triangular and rectangular finite elements in analysis of deep beam. *Academic Research International*, *3*(3), 131.

Fallah, N. A., Bailey, C., Cross, M., & Taylor, G. A. (2000). Comparison of finite element and finite volume methods application in geometrically nonlinear stress analysis. *Applied Mathematical Modelling*, 24(7), 439-455.

Ferguson, F., Mendez, J., Dodoo-Amoo, D., & Dhanasar, M. (2018). The performance evaluation of an improved finite volume method that solves the fluid dynamic equations. In 2018 AIAA Aerospace Sciences Meeting (p. 0834).

Fish, J. (Ed.). (2010). *Multiscale methods: bridging the scales in science and engineering*. Oxford University Press on Demand.

Foulkes, W. M. C., Mitas, L., Needs, R. J., & Rajagopal, G. (2001). Quantum Monte Carlo simulations of solids. *Reviews of Modern Physics*, *73*(1), 33.

Funken, S. A., & Schmidt, A. (2020). Adaptive mesh refinement in 2D–An efficient implementation in matlab. *Computational Methods in Applied Mathematics*, 20(3), 459-479.

Gao, W., & Liu, R. (2009). A hybrid finite volume/finite element method for incompressible generalized Newtonian fluid flows on unstructured triangular meshes. *Acta Mechanica Sinica*, 25(6), 747.

Gasca, M., & Sauer, T. (2001). On the history of multivariate polynomial interpolation. In *Numerical Analysis: Historical Developments in the 20th Century* (pp. 135-147). Elsevier.

Ghosh, S. (2015). Adaptive hierarchical-concurrent multiscale modeling of ductile failure in heterogeneous metallic materials. *Jom*, *67*(1), 129-142.

Godoy, L. A., & Dardatti, P.M. (2001). Validation of models in computational mechanics. *Computational Mechanics*, (17), 663-670.

Greengard, L., & Rokhlin, V. (1987). A fast algorithm for particle simulations. *Journal of computational physics*, 73(2), 325-348.

Gresho, P. M., & Lee, R. L. (1981). Don't suppress the wiggles—they're telling you something! Computers & Fluids, 9(2), 223-253.

Grote, M. J., Schneebeli, A., & Schötzau, D. (2006). Discontinuous Galerkin finite element method for the wave equation. *SIAM Journal on Numerical Analysis*, *44*(6), 2408-2431.

Gui, W. Z., & Babuška, I. (1986). The h, p and hp versions of the finite element method in 1 dimension. *Numerische Mathematik*, *49*(6), 613-657.

Guo, B., & Babuška, I. (1986). The hp version of the finite element method. *Computational Mechanics*, *1*(1), 21-41.

Hågbo, T. O., Giljarhus, K. E. T., Qu, S., & Hjertager, B. H. (2019, November). The performance of structured and unstructured grids on wind simulations around a high-rise building. In *IOP Conference Series: Materials Science and Engineering* (Vol. 700, No. 1, p. 012001). IOP Publishing.

Hamann, T., Qiu, G., & Grabe, J. (2015). Application of a Coupled Eulerian–Lagrangian approach on pile installation problems under partially drained conditions. *Computers and Geotechnics*, *63*, 279-290.

Hastings, W. F., Mathew, P., & Oxley, P. L. B. (1980). A Machining Theory for Predicting Chip Geometry, Cutting Forces, etc., from Work Material Properties and Cutting Conditions. Proc. Royal Soc. London, Ser. A, 371, 569-587.

He, B., & Zhuang, X. (2018). Modeling hydraulic cracks and inclusion interaction using XFEM. *Underground space*, 3(3), 218-228.

He, C., Ge, J., Zhang, B., Gao, J., Zhong, S., Liu, W. K., & Fang, D. (2020). A hierarchical multiscale model for the elastic-plastic damage behavior of 3D braided composites at high temperature. *Composites Science and Technology*, *196*, 108230.

He, X., & Ren, L. (2005). Finite volume multiscale finite element method for solving the groundwater flow problems in heterogeneous porous media. *Water resources research*, 41(10).

Hohenberg, P., & Kohn, W. (1964). Inhomogeneous electron gas. *Physical review*, *136*(3B), B864.

Hollenbeck, H., Wen, C., Romero, R., Ahmed, T., Genckal, N., Shirodkar, N., ... & Cheng, S. (2020). Multiscale Modeling of Fracture in Epoxy/CNT Nanocomposites. *Bulletin of the American Physical Society*, *65*.

Hou, T. Y., & Wu, X. H. (1997). A multiscale finite element method for elliptic problems in composite materials and porous media. *Journal of computational physics*, 134(1), 169-189.

Hou, T. Y., Hwang, F. N., Liu, P., & Yao, C. C. (2017). An iteratively adaptive multi-scale finite element method for elliptic PDEs with rough coefficients. *Journal of Computational Physics*, *336*, 375-400.

Hrennikoff, A. (1941). Solution of problems of elasticity by the framework method. *Journal of applied mechanics.* 8(4), 169-175.

Hu, Y., Ji, C., Zhang, Q., Chen, L., Zhan, P., & Li, X. (2020). A novel multi-resolution representation for time series sensor data analysis. *Soft Computing*, 24(14), 10535-10560.

Hwang, F. N., Su, Y. Z., & Yao, C. C. (2018). An iteratively adaptive multiscale finite element method for elliptic interface problems. *Applied Numerical Mathematics*, 127, 211-225.

Hyndman, R. J., & Fan, Y. (1996). Sample quantiles in statistical packages. *The American Statistician*, *50*(4), 361-365.

Iqbal, M., Gimperlein, H., Mohamed, M. S., & Laghrouche, O. (2017). An a posteriori error estimate for the generalized finite element method for transient heat diffusion problems. *International Journal for Numerical Methods in Engineering*, 110(12), 1103-1118.

Isakov, S. V., Mazzola, G., Smelyanskiy, V. N., Jiang, Z., Boixo, S., Neven, H., & Troyer, M. (2016). Understanding quantum tunneling through quantum Monte Carlo simulations. *Physical review letters*, *117*(18), 180402.

Jauregui Tellería, R., Rojas, J., & Silva Martínez, F. (2011). Study of transient phenomena with feature selective validation method. In *29th PIERS 2011 in Marrakesh, Morocco* (pp. 1113-1117).

Jebahi, M., André, D., Dau, F., Charles, J. L., & Iordanoff, I. (2013). Simulation of Vickers indentation of silica glass. *Journal of Non-Crystalline Solids*, *378*, 15-24.

Jebahi, M., Gakwaya, A., Lévesque, J., Mechri, O., & Ba, K. (2016). Robust methodology to simulate real shot peening process using discrete-continuum coupling method. *International Journal of Mechanical Sciences*, *107*, 21-33.

Jiang, S., & Du, C. (2017). Coupled finite volume methods and extended finite element methods for the dynamic crack propagation modelling with the pressurized crack surfaces. *Shock and Vibration*, 375130. Retrieved January 15, 2020, from https://www.hindawi.com/journals/sv/2017/3751340

Kamranian, M., Dehghan, M., & Tatari, M. (2017). An adaptive meshless local Petrov–Galerkin method based on a posteriori error estimation for the boundary layer problems. *Applied Numerical Mathematics*, *111*, 181-196.

Kaufmann, P. (2012). *Discontinuous Galerkin FEM in computer graphics* (Doctoral dissertation, ETH Zurich).

Kévin Pons, Mehmet Ersoy. (2019). Adaptive mesh refinement method. Part 1: Automatic thresholding based on a distribution function. Retrieved January 22, 2020, from https://hal.archives-ouvertes.fr/hal-01330679/file/PonsErsoyPart1_V2-3.pdf

Khattri, S. K. (2007). Grid generation and adaptation by functionals. *Computational & Applied Mathematics*, 26(2), 235-249.

Kiritsis, D., Emmanouilidis, C., Koronios, A., & Mathew, J. (Eds.). (2011). Engineering Asset Management: *Proceedings of the Fourth World Congress on Engineering Asset Management* (WCEAM) 2009. Springer Science & Business Media.

Kohnke, B., Kutzner, C., & Grubmüller, H. (2020). A GPU-accelerated Fast Multipole Method for GROMACS: performance and accuracy. *Journal of Chemical Theory and Computation*, 16(11), 6938-6949.

Lal, S. A., & Jabir, E. (2010). A hybrid finite element—finite volume method for incompressible flow through complex geometries using mixed grids. Proceedings of the Institution of Mechanical Engineers, Part G: *Journal of Aerospace Engineering*, 224(1), 23-41.

Lasis, A., & Süli, E. (2007). hp-version discontinuous Galerkin finite element method for semilinear parabolic problems. *SIAM journal on numerical analysis*, 45(4), 1544-1569.

Le Tallec, P., & Mallinger, F. (1997). Coupling Boltzmann and Navier–Stokes equations by half fluxes. *Journal of Computational Physics*, *136*(1), 51-67.

Li, J., Feng, X., & He, Y. (2019). RBF-based meshless local Petrov Galerkin method for the multi-dimensional convection–diffusion-reaction equation. *Engineering Analysis with Boundary Elements*, *98*, 46-53.

Li, L. Y., & Bettess, P. (1997). Adaptive finite element methods: a review. ASME. Appl. Mech. Rev, 50(10), 581–591.

Li, X. and Weinan, E. (2005). Multiscale modeling of the dynamics of solids at finite temperature. *Journal of the Mechanics and Physics of Solids*. 53(7), 1650-1685.

Li, X., & Dong, H. (2018). The element-free Galerkin method for the nonlinear p-Laplacian equation. *Computers & Mathematics with Applications*, *75*(7), 2549-2560.

Li, X., & Dong, H. (2019). Analysis of the element-free Galerkin method for Signorini problems. *Applied Mathematics and Computation*, *346*, 41-56.

Li, X., & Dong, H. (2021). An element-free Galerkin method for the obstacle problem. *Applied Mathematics Letters*, *112*, 106724.

Liew, S.C., (2017). Coupling of Adaptive Refinement with Variational Multiscale Element Free Galerkin Method for High Gradient Problems. PhD Thesis, Universiti Teknologi Malaysia, Skudai.

Lin, C., & Luo, K. H. (2018). Mesoscopic simulation of nonequilibrium detonation with discrete Boltzmann method. *Combustion and Flame*, *198*, 356-362.

Lingfa, K., Yidao, D., & Wei, L. (2020). The influence of global-direction stencil on gradient and high-order derivatives reconstruction of unstructured finite volume methods. *Acta Mechanica Sinica* , *52*(5), 1334-1349.

Liu, B., Sun, X., Bhattacharya, K., & Ortiz, M. (2021). Hierarchical multiscale quantification of material uncertainty. *Journal of the Mechanics and Physics of Solids*, *153*, 104492.

Liu, C., Shen, Z., Gan, L., Jin, T., Zhang, H., & Liu, D. (2018). A hybrid finite volume and extended finite element method for hydraulic fracturing with cohesive crack propagation in quasi-brittle materials. *Materials*, 11(10), 1921.

Liu, G. R., & Liu, M. B. (2003). *Smoothed particle hydrodynamics: a meshfree particle method*. World scientific Publishing Co. Pte. Ltd.

Liu, J., Qi, Y., Meng, Z. Y., & Fu, L. (2017). Self-learning monte carlo method. *Physical Review B*, *95*(4), 041101.

Liu, W.K., Karpov, E.G., Zhang, S. and Park, H.S. (2004). An introduction to computational nanomechanics and materials. *Computer Methods in Applied Mechanics and Engineering*. 193(17), 1529-1578.

Lumen Learning (2020). Fluid Dynamics and Its Biological and Medical Applications. Retrieved October 9, 2021, from https://courses.lumenlearning.com/waymaker-psychology/

Ma, Z., Korous, L., & Santiago, E. (2012). Solving a suite of NIST benchmark problems for adaptive FEM with the Hermes library. *Journal of Computational and Applied Mathematics*, 236(18), 4846-4861.

MacCormack, R., & Paullay, A. (1972, January). Computational efficiency achieved by time splitting of finite difference operators. In *10th Aerospace Sciences Meeting* (p. 154).

Madadi, H., Naghdinasab, M., & Farrokhabadi, A. (2020). Numerical investigation of matrix cracking propagation in cross-ply laminated composites subjected to three-point bending load using concurrent multiscale model. *Fatigue & Fracture of Engineering Materials & Structures*, *43*(6), 1159-1169.

Martin, T., Kamath, A., & Bihs, H. (2020). A Lagrangian approach for the coupled simulation of fixed net structures in a Eulerian fluid model. *Journal of Fluids and Structures*, *94*, 102962.

Martone, R., Formisano, A., Schober, M., & Kasper, M. (2007). Comparison of hp-adaptive methods in finite element electromagnetic wave propagation. *COMPEL-The international journal for computation and mathematics in electrical and electronic engineering*. 26(2), 431–446.

Maruyama, S. (2018). Molecular dynamics method for microscale heat transfer. In *Advances in numerical heat transfer* (pp. 189-226). CRC Press.

Mashayekhi, S., & Razzaghi, M. (2016). Numerical solution of distributed order fractional differential equations by hybrid functions. *Journal of Computational Physics*, 315, 169-181.

Matthai, S. K., Mezentsev, A. A., & Belayneh, M. (2007). Finite element-node-centered finite-volume two-phase-flow experiments with fractured rock represented by unstructured hybrid-element meshes. *SPE Reservoir Evaluation & Engineering*, 10(06), 740-756.

Mazzucato, A. L., Nistor, V., & Qu, Q. (2013). A nonconforming generalized finite element method for transmission problems. *SIAM Journal on Numerical Analysis*, 51(1), 555-576.

McDonald, P. W. (1971). *The computation of transonic flow through two-dimensional gas turbine cascades* (Vol. 79825, p. V001T01A089). American Society of Mechanical Engineers.

Mehrmann, V. (2010). Analysis and numerical solution of differential-algebraic equations with delay. Retrieved January 22, 2020, from https://www.depositonce.tu-berlin.de/bitstream/11303/4682/2/ha_phi.pdf

Metropolis, N., & Ulam, S. (1949). The monte carlo method. *Journal of the American statistical association*, *44*(247), 335-341.

Mitchell, W. F. (2013). A collection of 2D elliptic problems for testing adaptive grid refinement algorithms. *Applied mathematics and computation*, 220, 350-364.

Mitchell, W. F., & McClain, M. A. (2011). A survey of hp-adaptive strategies for elliptic partial differential equations. In *Recent advances in computational and applied mathematics* (pp. 227-258). Springer, Dordrecht.

Mitchell, W. F., & McClain, M. A. (2014). A comparison of hp-adaptive strategies for elliptic partial differential equations. *ACM Transactions on Mathematical Software (TOMS)*, 41(1), 1-39.

Msekh, M. A., Cuong, N. H., Zi, G., Areias, P., Zhuang, X., & Rabczuk, T. (2018). Fracture properties prediction of clay/epoxy nanocomposites with interphase zones using a phase field model. *Engineering Fracture Mechanics*, *188*, 287-299.

Muzhinji, K., Shateyi, S., & Motsa, S. S. (2015). The mixed finite element multigrid method for stokes equations. Retrieved January 20, 2020, from https://www.hindawi.com/journals/tswj/2015/460421/

Narumi, S. (1920). Some formulas in the theory of interpolation of many independent variables. *Tohoku Mathematical Journal*, *First Series*, 18, 309-321.

Nasiri, H., Jamalabadi, M. Y. A., Sadeghi, R., Safaei, M. R., Nguyen, T. K., & Shadloo, M. S. (2019). A smoothed particle hydrodynamics approach for numerical simulation of nano-fluid flows. *Journal of Thermal Analysis and Calorimetry*, *135*(3), 1733-1741.

Nave, C. (n.d). *Hyperphysics.* Retrieved October 8, 2021, from http://hyperphysics.phy-astr.gsu.edu/hphys.html

Nguyen, H. T. (2010). *p-adaptive and automatic hp-adaptive finite element methods for elliptic partial differential equations* (Doctoral dissertation, UC San Diego).

Nick, H. M., & Matthäi, S. K. (2011). A hybrid finite-element finite-volume method with embedded discontinuities for solute transport in heterogeneous media. *Vadose Zone Journal*, 10(1), 299-312.

Pal, B. and Ganesan, S. (2015). A finite element variational multiscale method for computations of turbulent flow over an aerofoil. *International Journal of Advances in Engineering Sciences and Applied Mathematics*, 7(1–2), 14–24.

Paluszny, A., Matthäi, S. K., & Hohmeyer, M. (2007). Hybrid finite element–finite volume discretization of complex geologic structures and a new simulation workflow demonstrated on fractured rocks. *Geofluids*, 7(2), 186-208.

Papadakis, G., & Voutsinas, S. G. (2019). A strongly coupled Eulerian Lagrangian method verified in 2D external compressible flows. *Computers & Fluids*, *195*, 104325.

Papadopoulos, P. (2010). Introduction to the Finite Element Method. *California: Berkeley University of California*.

Pardo, D., García-Castillo, L. E., Demkowicz, L. F., & Torres-Verdín, C. (2007). A two-dimensional self-adaptive hp finite element method for the characterization of waveguide discontinuities. Part II: Goal-oriented hp-adaptivity. *Computer methods in applied mechanics and engineering*, 196(49-52), 4811-4822.

Paszynski, M. A. C. I. E. J., & Pardo, D. (2011). Parallel self-adaptive hp finite element method with shared data structure. *Computer Methods in Materials Science*, 11(2), 399-405.

Payne, M. C., Teter, M. P., Allan, D. C., Arias, T. A., & Joannopoulos, A. J. (1992). Iterative minimization techniques for ab initio total-energy calculations: molecular dynamics and conjugate gradients. *Reviews of modern physics*, *64*(4), 1045.

Pfister, J. L., Marquet, O., & Carini, M. (2019). Linear stability analysis of strongly coupled fluid–structure problems with the Arbitrary-Lagrangian–Eulerian method. *Computer Methods in Applied Mechanics and Engineering*, *355*, 663-689.

Piastra, M. C., Nüßing, A., Vorwerk, J., Bornfleth, H., Oostenveld, R., Engwer, C., & Wolters, C. H. (2018). The discontinuous Galerkin finite element method for solving the MEG and the combined MEG/EEG forward problem. *Frontiers in Neuroscience*, 12, 30.

Pineda, E., Bednarcyk, B., Ricks, T., Arnold, S., & Henson, G. (2021). Efficient multiscale recursive micromechanics of composites for engineering applications. *International Journal for Multiscale Computational Engineering*.

Pons, K., & Ersoy, M. (2019). Adaptive mesh refinement method. Part 1: Automatic thresholding based on a distribution function. Retrieved January 10, 2020, from https://hal.archives-ouvertes.fr/hal-01330679v2

Prakash, C., & Patankar, S. V. (1985). A control volume-based finite-element method for solving the Navier-Stokes's equations using equal-order velocity-pressure interpolation. *Numerical heat transfer*, 8(3), 259-280.

Presho, M. (2018). Inverse modeling of tracer flow via a mass conservative generalized multiscale finite volume/element method and stochastic collocation. *Computational and Applied Mathematics*, 37(5), 6738-6759.

Presho, M., & Hill, M. (2021). A conservative generalized multiscale finite volume/element method for modeling two-phase flow with capillary pressure. *Journal of Computational and Applied Mathematics*, 381, 113026.

Profito, F. J., Giacopini, M., Zachariadis, D. C., & Dini, D. (2015). A general finite volume method for the solution of the Reynolds lubrication equation with a mass-conserving cavitation model. *Tribology Letters*, 60(1), 1-21.

Qin, X., Guo, J., Gu, C., Chen, X., & Xu, B. (2021). A discrete continuum coupled numerical method for fracturing behavior in concrete dams considering material heterogeneity. *Construction and Building Materials*, *305*, 124741.

Quesne, M. G., Borowski, T., & de Visser, S. P. (2016). Quantum mechanics/molecular mechanics modeling of enzymatic processes: Caveats and breakthroughs. *Chemistry–A European Journal*, *22*(8), 2562-2581.

Radu, F. A., Muntean, A., Pop, I. S., Suciu, N., & Kolditz, O. (2013). A mixed finite element discretization scheme for a concrete carbonation model with concentration-dependent porosity. *Journal of Computational and Applied Mathematics*, 246, 74-85.

Recktenwald, G. (2012). The control-volume finite-difference approximation to the diffusion equation. *Mechanical Engineering*, pp.1-18

Rice, J. R., Houstis, E. N., & Dyksen, W. R. (1981). A population of linear, second order, elliptic partial differential equations on rectangular domains. I, II. *Mathematics of Computation*, 36(154), 475-484.

Rodrigues, E. A., Gimenes, M., Bitencourt Jr, L. A., & Manzoli, O. L. (2021). A concurrent multiscale approach for modeling recycled aggregate concrete. *Construction and Building Materials*, *267*, 121040.

Rodrigues, E. A., Manzoli, O. L., Bitencourt Jr, L. A., Bittencourt, T. N., & Sánchez, M. (2018). An adaptive concurrent multiscale model for concrete based on coupling finite elements. *Computer Methods in Applied Mechanics and Engineering*, *328*, 26-46.

Ryabinkin, I. G., Yen, T. C., Genin, S. N., & Izmaylov, A. F. (2018). Qubit coupled cluster method: a systematic approach to quantum chemistry on a quantum computer. *Journal of chemical theory and computation*, *14*(12), 6317-6326.

Schwartzentruber, T.E. and Boyd, I.D. (2006). A hybrid particle-continuum method applied to shock waves. *Journal of Computational Physics*. 215(2), pp.402-416.

Sheikholeslami, M., & Ghasemi, A. (2018). Solidification heat transfer of nanofluid in existence of thermal radiation by means of FEM. *International Journal of Heat and Mass Transfer*, *123*, 418-431.

Shen, L., Cao, L. and Wong, Y. (2014). Multiscale Fem-Fvm Hybrid Method for Convection-Diffusion Equations with Periodic Discontinuous Coefficients in General Convex Domains. *International Journal of Numerical Analysis and Modeling, Series B*. 5(4), (pp.374–399).

Shi, C., Zhao, C., Zhang, X., & Guo, Y. (2020). Coupled discrete-continuum approach for railway ballast track and subgrade macro-meso analysis. *International Journal of Pavement Engineering*, 1-16.

Shi, D., Wang, J., & Yan, F. (2018). Unconditional superconvergence analysis of an H1-Galerkin mixed finite element method for nonlinear Sobolev equations. *Numerical Methods for Partial Differential Equations*, 34(1), 145-166.

Shurina, E. P., Solonenko, O. P., & Voitovich, T. V. (2002). Technologies of finite volume-finite element method for the solution of convection-diffusion problems on unstructured grids. Retrieved January 15, 2020, from https://cyberleninka.ru/article/n/technologies-of-finite-volume-finite-element-method-for-the-solution-of-convection-diffusion-problems-on-unstructured-grids

Siljak, E. (2009). Shape Functions generation, requirements, etc. Student presentation Ruhr Universität Bochum Fakultät für Bauingenieurwesen, p.46.

Retrieved January 25, 2020, from http://www.sd.ruhr-uni-bochum.de/downloads/Shape_funct.pdf

Singh, R., & Singh, K. M. (2019). Interpolating meshless local Petrov-Galerkin method for steady state heat conduction problem. *Engineering Analysis with Boundary Elements*, *101*, 56-66.

Sjodin, B. (2016). What's the difference between FEM, FDM, and FVM. *Machine Design*, 414.

Smith, G. D., Smith, G. D., & Smith, G. D. S. (1985). *Numerical solution of partial differential equations: finite difference methods*. Oxford university press.

Spiridonov, D., Vasilyeva, M., & Chung, E. T. (2020). Generalized Multiscale Finite Element method for multicontinua unsaturated flow problems in fractured porous media. *Journal of Computational and Applied Mathematics*, 370, 112594.

Steiner, T., Wriggers, P., & Loehnert, S. (2016). A discontinuous Galerkin Finite Element Method for linear elasticity using a mixed integration scheme to circumvent shear-locking. *PAMM*, 16(1), 769-770.

Stone, J. M., Tomida, K., White, C. J., & Felker, K. G. (2020). The Athena++ adaptive mesh Refinement framework: Design and magnetohydrodynamic solvers. *The Astrophysical Journal Supplement Series*, 249(1), 4.

Strang, G. and Fix, G.J. (1973). An analysis of the finite element method (Vol. 212). Englewood Cliffs, NJ: Prentice hall.

Struchkov, A. V., Kozelkov, A. S., Volkov, K. N., Kurkin, A. A., Zhuckov, R. N., & Sarazov, A. V. (2020). Numerical simulation of aerodynamic problems based on adaptive mesh refinement method. *Acta Astronautica*, 172, 7-15.

Szabo, A., & Ostlund, N. S. (2012). *Modern quantum chemistry: introduction to advanced electronic structure theory*. McGraw-Hill, New York. Courier Corporation.

Szabó, B., & Actis, R. (2011). FEM IN PROFESSIONAL PRACTICE: THE QUESTIONS OF 'WHAT?'AND 'HOW?'. In *Workshop on Higher Order Finite Element and Isogeometric Methods Program and Book of Abstracts* (p. 39).

Tao, H. (2018). A partitioned strong coupling algorith for fluid-structure interaction using arbitrary lagrangian-eulerian finite eleent forulation. *Chinese Journal of Theoretical and Applied Mechanics*, *50*(2), 395.

Teng, Z. H., Sun, F., Wu, S. C., Zhang, Z. B., Chen, T., & Liao, D. M. (2018). An adaptively refined XFEM with virtual node polygonal elements for dynamic crack problems. *Computational Mechanics,* 62(5), 1087-1106.

Tetik, T., Yildiz, R. A., Labanieh, A. R., Yoruk, B., Kursun Bahadir, S., Kalaoglu, F., & Koncar, V. (2021). Hydrodynamic modeling of e-textile fabric washing behavior by the Coupled Eulerian–Lagrangian method. *Textile Research Journal*, *91*(9-10), 1117-1131.

Thatte, A., & Salant, R. F. (2008). Hybrid Finite Element-Finite Volume Algorithm for Solving Transient Multi-Scale Non-Linear Fluid-Structure Interaction during Operation of a Hydraulic Seal. In *Proceedings of the COMSOL Conference*. Boston.

Valori, G., Pariat, E., Anfinogentov, S., Chen, F., Georgoulis, M. K., Guo, Y., ... & Yang, S. (2016). Magnetic helicity estimations in models and observations of the solar magnetic field. Part I: finite volume methods. *Space Science Reviews*, *201*(1), 147-200.

Ventosa Llopart, O., Jauregui Tellería, R., & Silva Martínez, F. (2011). Application of the Feature Selective Validation method to the analysis of transient phenomena. In *XXVI URSI Symposium* (pp. 1-4).

Vorwerk, J., Engwer, C., Pursiainen, S., & Wolters, C. H. (2016). A mixed finite element method to solve the EEG forward problem. *IEEE transactions on medical imaging*, 36(4), 930-941.

Vujanović, M., Petranović, Z., Edelbauer, W., & Duić, N. (2016). Modelling spray and combustion processes in diesel engine by using the coupled Eulerian–Eulerian and Eulerian–Lagrangian method. *Energy conversion and management*, *125*, 15-25.

Wadsley, J. W., Keller, B. W., & Quinn, T. R. (2017). Gasoline2: a modern smoothed particle hydrodynamics code. *Monthly Notices of the Royal Astronomical Society*, *471*(2), 2357-2369.

Wan, T., Aliabadi, S., & Bigler, C. (2009). Hybrid Finite Element and Finite Volume Methods for Two Immiscible Fluid Flows. In *19th AIAA Computational Fluid Dynamics* (p. 3887).

Wang, T., Yang, C., & Xie, X. (2019). Extended finite element methods for optimal control problems governed by Poisson equation in non-convex domains. *Science China Mathematics*, 1-18.

Wang, W., Wu, Y., Wu, H., Yang, C., & Feng, Q. (2021). Numerical analysis of dynamic compaction using FEM-SPH coupling method. *Soil Dynamics and Earthquake Engineering*, *140*, 106420.

Weinan, E., & Lu, J. (2011). Multiscale modeling. *Scholarpedia*, *6*(10), 11527.

Wen, J., He, Y. and Zhao, X. (2016). Analysis of a new stabilized finite volume element method based on multiscale enrichment for the Navier-Stokes problem. *International Journal of Numerical Methods for Heat & Fluid Flow*, 26(8), 2462-2485.

Widlund, O., & Toselli, A. (2004). Domain decomposition methods-algorithms and theory. In *Computational mathematics*. Springer.

Wördenweber, B. (1984). Finite element mesh generation. *Computer-aided design*, 16(5), 285-291.

Wu, C. T., Ma, N., Guo, Y., Hu, W., Takada, K., Okada, H., & Saito, K. (2018). A dynamic ductile failure analysis of shell structures using a nonlocal XFEM method with experimental validation. *Advances in Engineering Software*, 123, 1-12.

Xiao, S., Hu, R., Li, Z., Attarian, S., Björk, K. M., & Lendasse, A. (2020). A machine-learning-enhanced hierarchical multiscale method for bridging from molecular dynamics to continua. *Neural Computing and Applications*, *32*(18), 14359-14373.

Xie, Y., Lu, C., Xue, Y., Ye, Y., Xie, C., Ji, H., & Wu, J. (2019). New finite volume multiscale finite element model for simultaneously solving groundwater flow and darcian velocity fields in porous media. *Journal of Hydrology*, 573, 592-606.

Xu, H., Cantwell, C. D., Monteserin, C., Eskilsson, C., Engsig-Karup, A. P., & Sherwin, S. J. (2018). Spectral/hp element methods: Recent developments, applications, and perspectives. *Journal of Hydrodynamics*, 30(1), 1-22.

Xu, R., Yang, J., Yan, W., Huang, Q., Giunta, G., Belouettar, S., ... & Hu, H. (2020). Data-driven multiscale finite element method: From concurrence to separation. *Computer Methods in Applied Mechanics and Engineering*, 363, 112893.

Xu, T., Fu, T. F., Heap, M. J., Meredith, P. G., Mitchell, T. M., & Baud, P. (2020). Mesoscopic damage and fracturing of heterogeneous brittle rocks based on three-dimensional polycrystalline discrete element method. *Rock Mechanics and Rock Engineering*, *53*(12), 5389-5409.

Xue, B., Peng, Y. X., Ren, S. F., Liu, N. N., & Zhang, Q. (2021). Investigation of impact resistance performance of pyramid lattice sandwich structure based on SPH-FEM. *Composite Structures*, *261*, 113561.

Yamashita, H., Hart, R., Sharma, T., Samanta, A., Wang, Q., & Xiao, S. (2016). A review of multiscale methods and their applications in modeling and simulation of engineering problems. *Int. J. Recent Technol. Mech. Electr. Eng*, *3*(3), 42-47.

Yang, G., Wang, G., Lu, W., Yan, P., Chen, M., & Wu, X. (2018). A SPH-Lagrangian-Eulerian approach for the simulation of concrete gravity dams under combined effects of penetration and explosion. *KSCE Journal of Civil Engineering*, *22*(8), 3085-3101.

Yang, J., Wang, X., Zou, H., & Liang, G. (2010, December). A Combined Finite-Element and Finite-Volume Method in Reservoir Simulation. In *2010 Second World Congress on Software Engineering* (Vol. 2, pp. 325-328). IEEE.

Yokus, A., & Bulut, H. (2019). On the numerical investigations to the Cahn-Allen equation by using finite difference method. *An International Journal of Optimization and Control: Theories & Applications (IJOCTA)*, *9*(1), 18-23.

Zaid, M., Sadique, M. R., & Alam, M. M. (2021). Blast analysis of tunnels in Manhattan-Schist and Quartz-Schist using coupled-Eulerian–Lagrangian method. *Innovative Infrastructure Solutions*, *6*(2), 1-10.

Zeng, Q., & Qin, Y. (2018). Multiscale modelling of hybrid machining processes. In *Hybrid Machining: Theory, Methods, and Case Studies* (pp. 269-298). Academic Press.

Zhang, C. F., Wang, W., An, S. G., & Shentu, N. Y. (2021). Two-dimensional finite element mesh generation algorithm for electromagnetic field calculation. *Chinese Physics B*, 30(1), 010101.

Zhang, H., & Wang, A. (2018). A new approach of superconvergence analysis of a low order nonconforming MFEM for reaction–diffusion equation. *Boundary Value Problems*, 2018(1), 1-20.

Zhang, Y., Lee, T.S. and Yang, W. (1999). A pseudobond approach to combining quantum mechanical and molecular mechanical methods. *The Journal of chemical physics*, 110(1), 46-54.

Zheng, X., Yang, Z., Wang, S., Chen, Y. F., Hu, R., Zhao, X. J., ... & Yang, X. L. (2021). Evaluation of hydrogeological impact of tunnel engineering in a karst aquifer by coupled discrete-continuum numerical simulations. *Journal of Hydrology*, *597*, 125765.

Zhou, D., Liew, H. L., Purbolaksono, J., Andriyana, A., & Chong, W. T. (2019). Stress intensity factors for embedded cracks within torsionally loaded square prismatic bars. *Advances in Mechanical Engineering*, *11*(4), 1687814019828085.

Zienkiewicz, O. C., Morgan, K., & Morgan, K. (2006). Finite elements and approximation. Courier Corporation. Dover Publications, Inc. North Chelmsford.

LIST OF PUBLICATIONS

Olaiju, O. A., Hoe, Y. S., & Ogunbode, E. B. (2017). Finite difference approximations to the Heat Equation via C. *Journal of Applied Sciences and Environmental Sustainability*, *3*(7), 188-200.

Olaiju, O. A., Hoe, Y. S., & Ogunbode, E. B. (2018). Finite element and finite difference numerical simulation comparison for air pollution emission control to attain cleaner environment. *Chemical Engineering Transactions*, *63*, 679-684.

Olaiju, O. A., Hoe, Y. S., Ogunbode, E. B., Fabi, J. K., & Egba, E. I. (2018). Achieving a sustainable environment using numerical method for the solution of advection equation in fluid dynamics. *Chemical Engineering Transactions*, *63*, 631-636.