MATHEMATICAL ANALYSIS OF CASSON FLUID MODEL FOR DISPERSION OF SOLUTE IN BLOOD FLOW THROUGH A STENOSED ARTERY WITH CHEMICAL REACTION

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DEDICATION

To my beloved parents and my dear wife.

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ABSTRACT

The dispersion of solute in blood flow has gained great attention by scientists for long time due to its importance in medical field and drug manufacture. The main interest in this study is mathematical analysis for steady dispersion of solute in blood flow with chemical reaction through a stenosed artery. In this study, the blood is treating as a non-Newtonian of Casson fluid model. The momentum equation and constitutive equation of Casson fluid have been used to obtain the solutions of velocity and average velocity for stenosed artery. The Taylor-Aris technique is applied to get the distribution of concentration, effective axial diffusion and relative axial diffusion by considering chemical reaction. The solutions between the dispersion of solute through an artery with stenosis and without stenosis have been compared. It is found that the radius of stenosed artery decreases because of the stenosis length and stenosis height increases while in case of without stenosis the radius of the artery does not change. The effective axial diffusion decreases with the increase of yield stress of Casson fluid and chemical reaction. The relative axial diffusion decreases slightly with the increase of yield stress of Casson fluid. The increase of yield stress and Peclet number can increase the viscosity of fluid, therefore, it minimizes the relative axial diffusion. Finally, the relative axial diffusion decreases slowly with the increase of the yield stress while the relative axial diffusion increases slowly with the increase of the radius of stenosed artery.

ABSTRAK

Penyebaran bahan larut dalam aliran darah meraih banyak perhatian oleh saintis setelah sekian lama kerana kepentingannya dalam bidang perubatan dan pembuatan ubat. Kepentingan utama dalam kajian ini adalah analisis matematik untuk penyebaran mantap bahan larut dalam aliran darah dengan tindak balas kimia melalui arteri yang mempunyai stenosis. Dalam kajian ini, darah diandaikan sebagai model bendalir Casson tak Newtonan. Persamaan menakluk momentum dan persamaan juzuk bendalir Casson digunakan untuk memperoleh penyelesaian halaju dan halaju purata bagi arteri yang mempunyai stenosis. Teknik Taylor-Aris digunakan untuk mendapatkan pengagihan kepekatan, keresapan paksi berkesan dan keresapan paksi relatif dengan mempertimbangkan tindak balas kimia. Penyelesaian antara penyebaran bahan larut melalui arteri yang mempunyai stenosis dan tanpa stenosis telah dibandingkan. Hasilnya mendapati jejari arteri yang mempunyai stenosis berkurangan kerana saiz stenosis meningkat, manakala dalam kes tanpa stenosis, jejari arteri tidak berubah. Keresapan paksi berkesan berkurangan apabila tegasan alah bendalir Casson dan tindak balas kimia meningkat. Keresapan paksi relatif bagi bendalir Casson sedikit berkurangan dengan peningkatan tegasan alah. Peningkatan tegasan alah dan nombor Peclet boleh meningkatkan kelikatan bendalir, seterusnya meminimumkan keresapan paksi relatif. Akhirnya, keresapan paksi relatif berkurangan perlahan-lahan dengan peningkatan tegasan alah manakala keresapan paksi relatif meningkat perlahan-lahan dengan peningkatan jejari arteri yang mempunyai stenosis.

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LIST OF ABBREVIATIONS

CFM	-	Curve-Fitting Method
DLS	-	Duckworth–Lewis–Stern
GDM	-	Generalized Dispersion Model
H-B	-	Herschel-Bulkley
HPM	-	Homotopy Perturbation Method
K-L	-	Kuang-Luo
MAC	-	Marker and Cell Method
SOR	-	Successive Over Relaxation Method
SEC-MALS	-	Size-Exclusion Chromatography Coupled to Multi-Angle Light Scattering
TDA	-	Taylor Dispersion Analysis

LIST OF SYMBOLS

$\bar{D}_{eff}$	-	Effective axial diffusion
$\bar{D}_m$	-	Molecular diffusion
a	-	Radius of artery
$\bar{z}$	-	Axial coordinate for artery
$\bar{p}$	-	Pressure
$\bar{r}$	-	Radial coordinate for artery flow
r	-	Non-dimensional radial coordinate for artery flow
Re	-	Reynold Number
$\bar{R}(\bar{z})$	-	Radius of stenosed artery
$\bar{r}_c$	-	Radius of plug flow region
r_c	-	Non-dimensional radius of the plug flow region
$\bar{u}$	-	Axial velocity of the fluid flow
$\bar{u}_+$	-	Axial velocity in the outer flow region
$\bar{u}_-$	-	Axial velocity in the plug flow region
$\hat{u}_-$	-	Average velocity in plug region
$\hat{u}_+$	-	Average velocity in outer region
$\bar{u}_m$	-	Average velocity
$\bar{C}$	-	Concentration distribution of the solute.
$\bar{C}_1$	-	Concentration of the solute in the plug flow region.
$\bar{C}_2$	-	Concentration of the solute in the outer flow region
Pe	-	Peclet number
$\bar{q}$	-	Flux of solute
K	-	Chemical reaction rate parameter
l_0	-	Stenosis length
E / A^2	-	Impact of the velocity for the steady flow with and without chemical reaction in a stenosed artery

Greek symbols

$\bar{\tau}$	-	Shear stress
$\bar{\tau}_y$	-	Yield stress
$\bar{\eta}_c$	-	Viscosity coefficient of Casson fluid model
$\bar{\theta}$	-	Azimuthal angle
$\bar{\gamma}$	-	Shear rate
δ	-	Stenosis height
$\bar{\rho}$	-	Density of the fluid

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CHAPTER 1

INTRODUCTION

1.1 Introduction

In this chapter, the study discusses the problem background regarding the dispersion of solute, Casson fluid model, stenosis and chemical reaction. There are several impact factors into the effectiveness of the dispersion such as the stenosis size (the stenosis height and length), the chemical reaction, yield stress and Peclet number. In this study, the stenosis size and the chemical reaction are considered through a stenosed artery.

Nowadays, the researchers pay many attentions on the dispersion of solute in blood flow. Hence the mathematical analysis of solute dispersion in blood flow becomes very important in the determining the effective solute dispersion coefficient, effective concentration of solute, effective time and effective fluid model of blood. The steady state of solute dispersion in blood flow is an important part to know the blood rheology in narrow arteries and to obtain result that useful in investigating some physical problems involving dispersion of solute in blood flow. The steady dispersion with chemical reaction through a stenosed artery needs constant conditions and properties (condition and properties should remain unchanged through the artery) for happen the process with time at different points. Water is an example of steady flow that being pumped within a stable system at a constant rate (Swarup, 2000).

The arteries are part of vessel which can be educed disease called arteriosclerosis. The arteriosclerosis also known as stenosis. This disease is caused by the accumulation of fat, cells and other materials. When this arteriosclerosis happens, blood vessels suffer from tightness which is the blood flow become more difficult as shown in Figure 1.1. Figure 1.1 illustrates the type of arteries normal and artery with plaque build. This is referred to the arteriosclerosis, which more affects blood vessels that transport blood to the brain, heart and legs most often (Sharma *et al*., 2015). Narrowing of blood vessels may affects both genders, although arteriosclerosis can take decades to appear, but it is not usually detected until men reach 40 years of age and women until their 50s and 60s (Aronow *et al*., 2013). Neural arteriosclerosis does not usually have display until the distress is severe. A lot of patients do not sense that they have narrowed blood vessels even after a stroke. Signs may also appeared vary depending on the artery or member most affected. The diseases such as nausea, dizziness, blurred vision and severe headaches maybe indicator to start arteriosclerosis. When arteriosclerosis becomes severe (usually after some diseases that have been detected like a stroke or heart attack. The neurosurgeon may use minimally direct surgery to remove or expand the stenosed artery (Aronow *et al*., 2013).

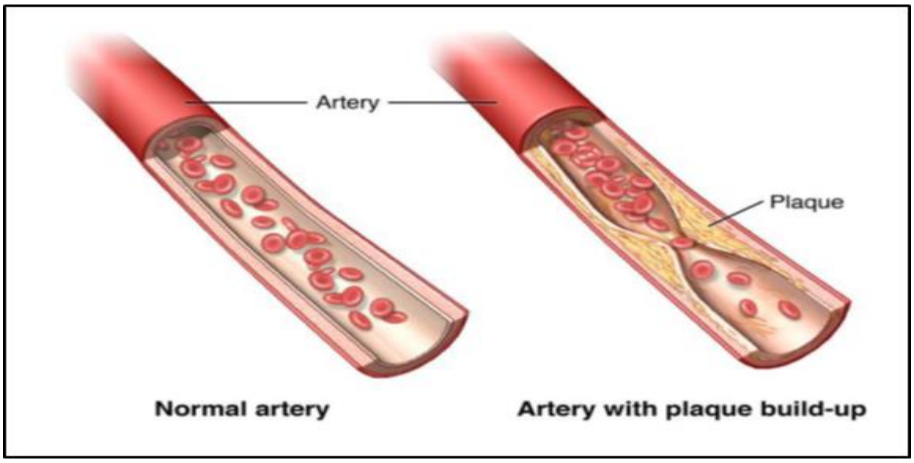


Figure 1.1 Normal artery and artery with plaque build-up

Many vast amounts of research have been conducted to better understanding of fluid behavior due to numerous types of fluids that usually encountered in industry and daily life which are known as Newtonian and non-Newtonian. The Newtonian fluid (called Isaac Newton) is a fluid that is stress-resistant versus the linear stress curve and passes through the origin. The constant proportionality is known as the viscosity (Karol, 2003; Krizek and Pepper, 2004). Non-Newtonian fluid is a fluid which is flow properties are not described by a fixed value of viscosity. Numerous mathematical models were proposed for fluid behavior that can be model the properties of shear thinning. These models depend on the curve-fitting method (CFM) which gives empirical relations to the curves of the shear stress curve versus the shear curves.

For example, the Bingham fluid model does not accurately describes the behavior of cement-based plaster even if the geometry is determined, but there are other models can describe it exactly. Thus, the model must be chosen carefully based on the case that want to describe. Thus, Casson fluid model is one of the best mathematical models used to describe the rheology of blood flow in narrow arteries at moderate shear rate (Astarita and Marrucci, 1974)

The constitutive equation of Casson model is the relationship that allows calculation of pressure as a function of the kinematic variables and ultimately as a function in the field of velocity that may depend on time (Astarita and Marrucci, 1974). The viscosity, yield stress and shear stress play an important role in the constitutive equations. The constitutive equation of Casson model consists of the yield stress as given by

$$\bar{\gamma} = \begin{cases} \dfrac{1}{\bar{\eta}_c}\left(\sqrt{\bar{\tau}} - \sqrt{\bar{\tau}_y}\right)^2 & \text{when} \quad \bar{\tau} \ge \bar{\tau}_y , \\ 0 & \text{when} \quad \bar{\tau} < \bar{\tau}_y , \end{cases} \tag{1.1}$$

where $\bar{\eta}_c = \bar{\mu} - 2\sqrt{\bar{\tau}\,\bar{\tau}_y} / \bar{\gamma} + \bar{\tau}_y / \bar{\gamma}$ is the viscosity coefficient of Casson fluid with $ML^{-1}T^{-1}$ dimension, $\bar{\tau}_y$ is the yield stress. When $\bar{\tau}_y = 0$, the Casson fluid model reduces to the Newtonian fluid model. In addition, this model can be used only for moderate shear rate in smaller diameter tubes. Many researchers such as Blair (1959) and Copley (1960) modelled Casson fluid in the blood flow in narrow arteries at low shear rates. They demonstrated that the Casson fluid model is satisfactory for the description of the simple shear behavior of blood in narrow arteries.

Taylor-Aris technique was the first method to measure the diffusion coefficients of molecules in steady dispersion of solute. This method described first time in 1953 by Taylor when he measured the dispersion of a pulse within a capillary. This method extended by computing the longitudinal molecules of diffusion by Aris (1956). The main idea of the Taylor-Aris technique is computing the concentration distribution of solute by assuming a symmetric Gaussian distribution which moves downstream with the average velocity of the flow and spreads in the flow direction with an effective diffusion coefficient $\bar{D}_{eff}$ which is larger than the molecular diffusion coefficient $\bar{D}_m$ because of the existence of the flow. This phenomenon is now defined as Taylor-Aris technique or Taylor dispersion (Cottet *et al*., 2007). There are other methods to analysis the concentration distribution such as duckworth–lewis–stern (DLS) and size-exclusion chromatography coupled to multi-angle light scattering (SEC-MALS) but have not been used as Taylor-Aris technique in the field to date (Malvern, 2015). Taylor-Aris technique is considered as fast and absolute method for determining the effective diffusion coefficient from 1950s (Cottet *et al*., 2007). Taylor-Aris technique is also namely as Taylor dispersion analysis (TDA) (Malvern, 2015). This study analyzes the concentration distribution, effective axial diffusion and relative axial diffusion by using Taylor-Aris technique to obtain the accuracy solutions.

In this study, the construction of analytically technique which is integration has been considered to solve the momentum and constitutive equations. In addition, the Simpson rule and Bessel function has been considered to solve convective-diffusion equation. *Mathematica* has been used to obtain the data for concentration distribution, effective axial diffusion and relative axial diffusion in case of with and without chemical reaction. Taylor-Aris technique are applied to obtain the effective axial diffusion and relative axial diffusion. The analytical solutions have been analyzed, and described the steady dispersion process in a stenosed artery.

1.2 Problem Statement

Many researchers only treated the blood as a Newtonian fluid where it flows in arteries with large diameter at high shear rate but it is also important to study the blood flow in a narrow artery with moderate shear rates. Thus, this study analyze the dispersion of solute in blood flow through a stenosed artery with and without the presence of chemical reaction to obtain the data. Thus, using an appropriate fluid model such as Casson fluid model in a stenosed artery for moderate shear rate can produce more realistic results. The momentum and constitutive equations of Casson fluid have been used to obtain the solutions of velocity and average velocity for stenosed artery by integrating with suitable boundary conditions. The convective-diffusion equation has been solved analytically by using integration to account concentration distribution without chemical reaction. While the convective-diffusion with chemical reaction for steady dispersion has been solved numerically using *Mathematica* because complexed function of Bessel function. The Taylor-Aris technique is applied to get distribution of concentration, flux of solute, effective axial diffusion and relative axial diffusion with and without chemical reaction. In this research, the impact of the stenosis size and chemical reaction on the dispersion process in blood flow also has to take into account to make sure that the solutions obtained will be more realistic and the medicine that will be created will be more efficient when given to the patients. To the writer's knowledge, no researcher attempted the study on the impact of stenosis size through a stenosed artery with chemical reaction in the steady flow in blood flow by treating the blood as Casson fluid.

1.3 Research Objectives

The aims of this study are:

1. To formulate mathematical model of Casson fluid model in a stenosed artery.
2. To solve analytically the momentum and constitutive equations for the steady dispersion of solute within a stenosed artery using integration with a suitable boundary condition.
3. To solve the steady convective-diffusion equations without chemical using integration and with chemical reaction using Bessel function.
4. To analyze the dispersion of solute by determining the expression of concentration, flow of solute, effective axial diffusion and relative axial diffusion using *Mathematica*.

1.4 Scope of Study

The scope of this study is restricted to problems related axisymmetric, steady, laminar, and fully developed unidirectional flows of viscous incompressible Casson fluid in the axial direction through a stenosed artery. The governing equations of momentum and constitutive equations are solved analytically to get the velocity and average velocity in a stenosed artery. The velocity and average velocity are used in the steady convective-diffusion equation to get the concentration of solute in both flow regions. The Taylor-Aris technique is applied in this problem to obtain concentration distribution, effective axial diffusion and relative axial diffusion with and without chemical reaction. The results of solute dispersion in Casson fluid model can be reduced to get the result of Newtonian. The comparison between the dispersion of solute with stenosis and without stenosis has been compared. In addition, the impact of chemical reaction has been investigated.

1.5 Significance of Study

The study of solute dispersion in blood flow through a stenosed artery can help doctors, medical lab technicians or physiologists in predicting the suitable amount of medicine to be given to the patients. The findings also can help the other researcher especially in medical, pharmaceutical and bioengineering fields to get the appropriate assumptions and realistic descriptions dispersion of solute in blood flow when handling many cardiovascular diseases. Thus, the significant of this study are:

1. This study determines the obtained result which are velocity and average velocity by solving analytically the momentum and constitutive equations using integration.
2. This study determines the expression of concentration, effective axial diffusion and relative axial diffusion by solving analytically the convective-diffusion equation with and without chemical reaction using Taylor-Aris technique.
3. This study also considers the stenosis size (stenosis length and height) with and without chemical reaction in the stenosed artery, and the results can be compared to the solute dispersion without stenosis.

1.6 Organization of Dissertation

This dissertation consists five chapters. Chapter 1 introduces the main notions of dispersion of solute such as fluid models, the arteriosclerosis, fluid mechanics, governing equations and objective of the research. In Chapter 2, the related literature reviews are presented. Chapter 3 discusses the research methodology and provides the analytical solution of velocity, average velocity, distribution concentration, effective axial diffusion and relative axial diffusion. The analysis results and comparisons using different values of the parameters are analyzed by graphs and tables in Chapter 4. Finally, Chapter 5 gives conclusion and some recommendation for future studies.

REFERENCES

Akbar, N.S., Hayat, T., Nadeem, S., Hendi, A.A. 2013. Influence of mixed convection on blood flow of Jeffrey fluid through a tapered stenosed artery. *Thermal Science*, **17**(2), 533-546.

Akbar, N.S., Nadeem, S., Mekheimer, K.S. 2016. Rheological properties of Reiner-Rivlin fluid model for blood flow through tapered artery with stenosis. *Journal of the Egyptian Mathematical Society*, **24**(1), 138-142.

Ali, N., Zaman, A., Sajid, M., Nieto, J., Torres, A. 2015. Unsteady non-Newtonian blood flow through a tapered overlapping stenosed catheterized vessel. *Mathematical Biosciences*, **269**, 94-103.

Ananthakrishnan, V., Gill, W.N., Barduhn, A.J. 1965. Laminar dispersion in capillaries: Part I. Mathematical analysis. *AIChE Journal*, **11**(6), 1063-1072.

Aris, R. 1956. On the dispersion of a solute in a fluid flowing through a tube. *Proceedings of the Royal Society of London. Series A. Mathematical and Physical Sciences*, **235**(1200), 67-77.

Aronow, W.S., Fleg, J.L., Rich, M.W. 2013. *Tresch and Aronow's Cardiovascular Disease in the Elderly*. CRC Press.13: 978-1-8421-543-2

Astarita, G. 1974. G., Marrucci. *Principles of Non-Newtonian Fluid Mechanics, McGraw-Hill, London, England*.

Bakheet, A., Alnussaiyri, E.A., Ismail, Z. and Amin, N., 2016. Blood flow through an inclined stenosed artery, *Applied Mathematical Sciences*, **10(**5), 235-254.

Barton, N. 1983. On the method of moments for solute dispersion. *Journal of Fluid Mechanics*, **126**, 205-218.

Blair, G. S. 1959. An equation for the flow of blood, plasma and serum through glass capillaries. *Nature*, **183**(4661), 613-614.

Bird, R.B., Stewart, W.E., Lightfoot, E.N. 1960. Transport phenomena John Wiley & Sons, *New York* **413**.

Cengel, Y., Cimbala, J. 2006. Obstruction Flowmeters: Orifice, Venturi, and Nozzle Meters. *Fluid Mechanics: Fundamentals and Applications, McGraw-Hill, New York,* 382-383.

Chitra, M., Karthikeyan, D. 2018. Mathematical modeling of power law and Herschel-Buckley non-Newtonian fluid of blood flow through a stenosed artery with permeable wall: Effects of slip velocity. *Journal of Physics: Conference Series*. **1000** (2018) 012064.

Copley, A. L. 1960. Apparent viscosity and wall adherence of blood systems, in Flow Properties of Blood and Other Biological Systems. *Ed. Copley, A. L. and Stainsly, G.; Oxford UK: Pergamon Press.* **96**(8), 3549-3560.

Cottet, H., Biron, J.-P., Martin, M. 2007. Taylor dispersion analysis of mixtures. *Analytical Chemistry*, **79**(23), 9066-9073.

Decuzzi, P., Causa, F., Ferrari, M., Netti, P. 2006. The effective dispersion of nanovectors within the tumor microvasculature. *Annals of Biomedical Engineering*, **34**(4), 633-641.

Garcia, A.E., Riahi, D.N. 2013. The Two-Phase Arterial Blood Flow with or without a Catheter and in the Presence of a Single or Multi Stenosis. *Founders and Founding Editors-in-Chief*, **8**(2), 506-522.

Gentile, F., Ferrari, M., Decuzzi, P. 2007. Transient Diffusion of Nanovectors in Permeable Capillaries. *Journal of the Serbian Society for Computational Mechanics*, **1**(1), 1-19.

Gentile, F., Ferrari, M., Decuzzi, P. 2008. The transport of nanoparticles in blood vessels: the effect of vessel permeability and blood rheology. *Annals of Biomedical Engineering*, **36**(2), 254-261.

Gill, W.N. 1967. A note on the solution of transient dispersion problems. *Proceedings of the Royal Society of London. Series A. Mathematical and Physical Sciences*, **298**(1454), 335-339.

Gill, W.N., Sankarasubramanian, R., Taylor, G.I. 1971. Dispersion of a non-uniform slug in time-dependent flow. *Proceedings of the Royal Society of London. A. Mathematical and Physical Sciences*, **322**(1548), 101-117.

Gill, W., Sankarasubramanian, R. 1970. Exact analysis of unsteady convective diffusion. *Proceedings of the Royal Society of London. A. Mathematical and Physical Sciences*, **316**(1526), 341-350.

Gupta, P., Gupta, A. 1972. Effect of homogeneous and heterogeneous reactions on the dispersion of a solute in the laminar flow between two plates. *Proceedings of the Royal Society of London. A. Mathematical and Physical Sciences*, **330**(1580), 59-63.

Jaafar, N.A., Yatim, Y.M., Sankar, D. 2016. Mathematical analysis for unsteady dispersion of solute with chemical reaction in blood flow. *AIP Conference Proceedings*, **1750**(1),030033.

Jaafar, N.A., Yatim, Y.M., Sankar, D. 2017. Effect of chemical reaction in solute dispersion In Herschel-Bulkley fluid flow with applications to blood flow. *Advances and Applications in Fluid Mechanics,* **20**(2) 0973-4686, 279-310.

Jahangiri, M., Saghafian, M., & Sadeghi, M. R. 2015. Numerical simulation of hemodynamic parameters of turbulent and pulsatile blood flow in flexible artery with single and double stenoses. *Journal of Mechanical Science and Technology*, **29**(8), 3549-3560.

Johnson, M., Kamm, R.D. 1986. Numerical studies of steady flow dispersion at low Dean number in a gently curving tube. *Journal of Fluid Mechanics*, **172**, 329-345.

Karol R.H. 2003. Chemical grouting and soil stabilization. 3 st ed., *Marcel Dekker Inc*., New Jersy, USA, **12**, 0-203-91181, 1-558.

Krizek R.J., Pepper S.F 2004. Slurries in geotechnical engineering, *Texas A&M University*, Texas, USA, 1-5.

Copley, L. Apparent viscosity and wall adherence of blood systems, 1960. *In Flow Properties of Blood and Other Biological Systems, A. L. Copley and G. Stainsly, Eds., Pergamon Press*, Oxford, UK.

Malek, A., Hoque, A., Mohiuddin, M. 2015. Effect of Hematocrit Level on the Blood Flow t through Stenosed Artery: A Theoretical Study. *Engineering International*, **3**(2), 87-96.

Malvern. 2015. Understanding Taylor Dispersion Analysis. *Malvern Instruments Limited*. Grovewood Road, Malvern, Worcestershire, UK. WR14 1XZ, pp. 1-8.

Mishra, S. and Siddiqui, S. U. 2011. A mathematical model for flow and diffusion through stenotic capillary-tissue exchange system, *E-Journal of Science and Technology*, **2**(6), 1-17.

Misra, J.C., Shit, G.C., Pramanik, R. 2018. Non-Newtonian flow of blood in a catheterized bifurcated stenosed artery. *Journal of Bionic Engineering*, **15**(1), 173-184.

Papanastasiou, Tasos C., Georgiou, G. and Alexandrou, A.N., 1999. *Viscous fluid flow*. CRC press. *Libaray of Congress Cataloging.* (**532**) 99-042163, 1-413

Patel, I., Sirs, J. 1983. Dispersion of solutes during blood flow through curved tubes. *Medical and Biological Engineering and Computing*, **21**(2), 113-118.

Rahman, S., Ellahi, R., Nadeem, S., Zia, Q.Z. 2016. Simultaneous effects of nanoparticles and slip on Jeffrey fluid through tapered artery with mild stenosis. *Journal of Molecular Liquids*, **218**, 0167-7322, 484-493.

Ratchagar, N.P., Kumar, R.V. 2019. Dispersion of solute with chemical reaction in blood flow. *Bulletin of Pure & Applied Sciences-Mathematics and Statistics*, **38**(1), 385-395.

Sankarasubramanian, R., Gill, W.N. 1973. Unsteady convective diffusion with interphase mass transfer. *Proceedings of the Royal Society of London. A. Mathematical and Physical Sciences*, **333**(1592), 115-132.

Shah, S. 2011. Capillary-tissue diffusion phenomena for blood flow through a stenosed artery using Herschel-Bulkley fluid. *International Journal of Biochemistry and Biophysics*, **1**, 1-8.

Sharma, P., Georgescu, B., Mihalef, V., Chen, T., Comaniciu, D. 2015. Method and system for non-invasive assessment of coronary artery disease, *Patent No.* 9,999,361.

Sharp, M.K. 1993. Shear-augmented dispersion in non-Newtonian fluids. *Annals of Biomedical Engineering*, **21**(4), 407-415.

Sharp, M.K., Kamm, R., Shapiro, A., Kimmel, E., Karniadakis, G. 1991. Dispersion in a curved tube during oscillatory flow. *Journal of Fluid Mechanics*, **223** (-1), 537-563.

Shaw, S., Ganguly, S., Sibanda, P., Chakraborty, S. 2014. Dispersion characteristics of blood during nanoparticle assisted drug delivery process through a permeable microvessel. *Microvascular Research*, **92** (2014), 25-33.

Shivakumar, P., Rudraiah, N., Pal, D., Siddheshwar, P. 1987. Closed form solution for unsteady convective diffusion in a fluid-saturated sparsely packed porous medium. *International Communications in Heat and Mass Transfer*, **14**(2), 137-145.

Sriyab, S. 2014. Mathematical analysis of non-Newtonian blood flow in stenosis narrow arteries. *Computational and Mathematical Methods in Medicine*, **2014**, 479152. doi:10.1155/2014/479152

Swarup, S. 2000. Fluid Dynamics. *Krishna Prakashan Media (P) Ltd. Merut*, **622**, 633.

Taylor, G.I. 1954a. Conditions under which dispersion of a solute in a stream of solvent can be used to measure molecular diffusion. *Proceedings of the Royal Society of London. Series A. Mathematical and Physical Sciences*, **225**(1163), 473-477.

Taylor, G.I. 1954b. The dispersion of matter in turbulent flow through a pipe. *Proceedings of the Royal Society of London. Series A. Mathematical and Physical Sciences*, **223**(1155), 446-468.

Taylor, G.I. 1953. Dispersion of soluble matter in solvent flowing slowly through a tube. *Proceedings of the Royal Society of London. Series A. Mathematical and Physical Sciences*, **219**(1137), 186-203.

Venkatesan, J., Sankar, D., Hemalatha, K., Yatim, Y. 2013. Mathematical analysis of Casson fluid model for blood rheology in stenosed narrow arteries. *Journal of Applied Mathematics*, **2013**(583809), 1-11.

APPENDIX A

THE CODING OF $\bar{C}_1$, $\bar{C}_2$ and *E* WITHOUT CHEMICAL REACTION

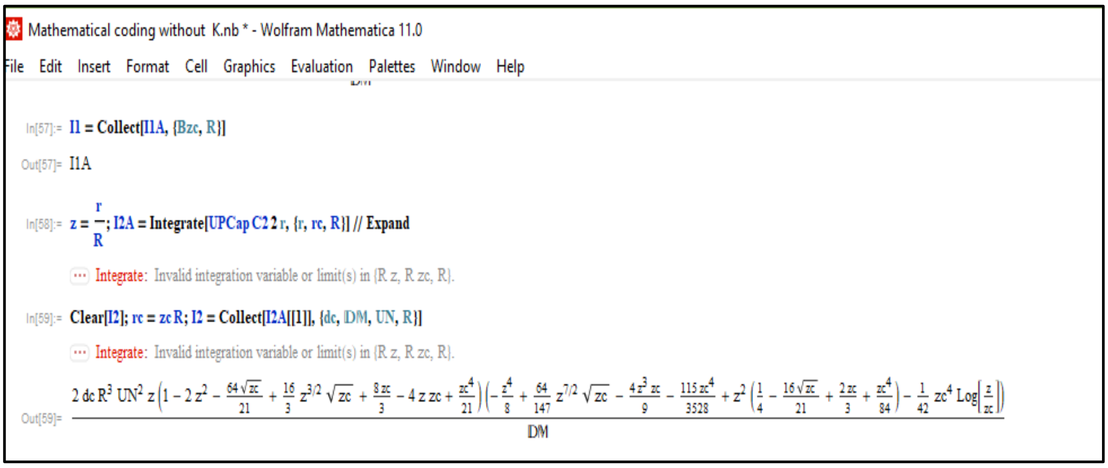
A.1 Coding of Concentration in Plug Region $\bar{C}_1$ and $\bar{C}_c$

```
Mathematical coding without K.nb * - Wolfram Mathematica 11.0
File  Edit  Insert  Format  Cell  Graphics  Evaluation  Palettes  Window  Help

In[38]:= Clear["Global`*"]; Clear[DM]; K = 0;
         Bzc = 1 - 64/21 zc^(1/2) + 8/3 zc - 2/3 zc^2 + 1/21 zc^4;
         Clear[Bzc];
         UMCap = UN Bzc;(*(UM=u_)*)
         CP = DSolve[{DM/r D[r C'[r], r] - UMCapdc - K C[r] == 0, C'[0] == 0, C[0] == 0}, C, r](*dc stand for dc/dz*)
Out[42]= {{C -> Function[{r}, (r^2 UMCapdc)/(4 DM)]}}

In[43]:= C1 = CP[[1, 1, 2, 2]]
Out[43]= (r^2 UMCapdc)/(4 DM)

In[44]:= r = rc;  Cc = C1
         Clear[r]
Out[44]= (rc^2 UMCapdc)/(4 DM)

In[46]:= Clear[r, rc]; Cc
Out[46]= (rc^2 UMCapdc)/(4 DM)
```

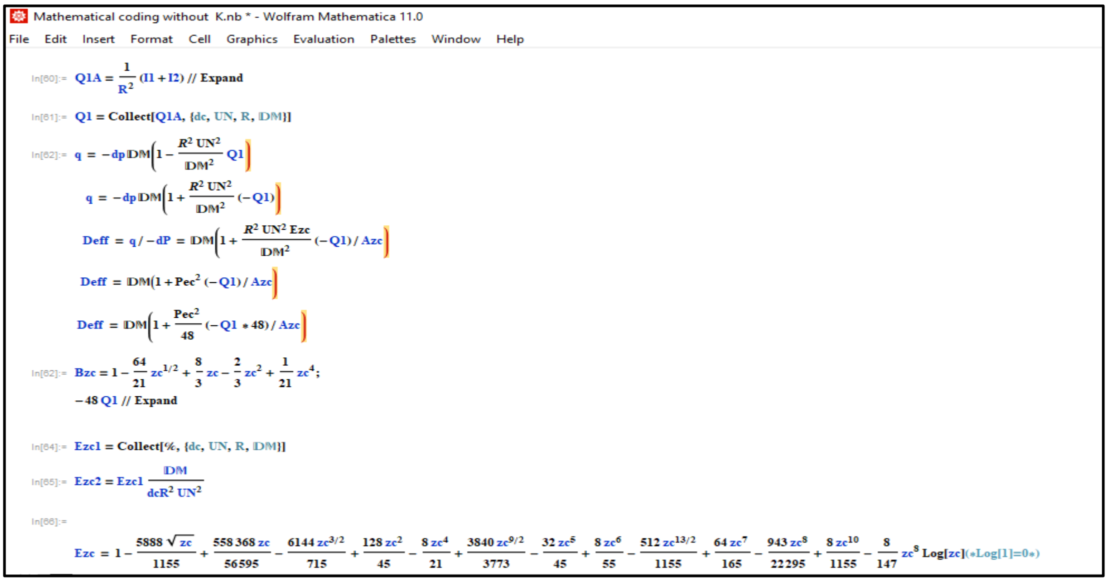
A.2 Coding of Concentration in Outer Region $\bar{C}_2$

```
Mathematical coding without K.nb * - Wolfram Mathematica 11.0
File  Edit  Insert  Format  Cell  Graphics  Evaluation  Palettes  Window  Help
In[47]:= Clear[z, zc, r, rc]; z = r / R;  Clear[CP2];
         UPCap = UN (1 - 2 z^2 - 64/21 zc^(1/2) + 8/3 zc + 1/21 zc^4 + 16/3 zc^(1/2) z^(3/2) - 4 zc z);(*(UP=u+)*)
         CP2 = DSolve[{DM/r C''[r] + C'[r] - r UPCapdc - r K C[r] == 0, C'[R] == 0, C[rc] == Cc}, C, r]

In[50]:= CP2A = CP2[[1, 1, 2, 2]] // Simplify

In[51]:= Clear[r, z, rc, zc];
         Bzc = 1 - 64/21 zc^(1/2) + 8/3 zc - 2/3 zc^2 + 1/21 zc^4;
         r = z R; rc = zc R;
         C2A = CP2A // ExpandAll

In[55]:= C2B = Collect[C2A, {dc/DM, UN, R^2, z^2}]

In[56]:= C2 = 1/DM dc UN R^2 (-z^4/8 + 64/147 z^(7/2) Sqrt[zc] - (4 z^3 zc)/9 - (115 zc^4)/3528 + z^2 (1/4 - (16 Sqrt[zc])/21 + (2 zc)/3 + zc^4/84) - 1/42 zc^4 Log[z/zc])
Out[56]= (dc R^2 UN (-z^4/8 + 64/147 z^(7/2) Sqrt[zc] - (4 z^3 zc)/9 - (115 zc^4)/3528 + z^2 (1/4 - (16 Sqrt[zc])/21 + (2 zc)/3 + zc^4/84) - 1/42 zc^4 Log[z/zc]))/DM
```

A.3 Coding of I_1 and I_2



A.4 Coding of q and E



APPENDIX B

THE CODING OF $\bar{C}_1$, $\bar{C}_c$, $\bar{C}_2$ and *E*

WITH CHEMICAL REACTION

B.1 Coding of Concentration in Plug Region $\bar{C}_1$ and $\bar{C}_c$

```
Final graphs ,table.nb * - Wolfram Mathematica 11.0
File  Edit  Insert  Format  Cell  Graphics  Evaluation  Palettes  Window  Help

C1 with chemical

Clear["Global`*"]; Clear[DM]; Clear[K];
Bzc = 1 - 64/21 zc^(1/2) + 8/3 zc - 2/3 zc^2 + 1/21 zc^4;
Clear[Bzc];
UMCap = UN Bzc;(*(UM=u_)*)
CP = DSolve[{DM/r D[r C'[r], r] - UMCapdc - K C[r] == 0, C'[0] == 0, C[0] == 0}, C, r](*dc stand for dc/dz*)

{{C -> Function[{r}, (-UMCapdc + UMCapdc BesselJ[0, (i Sqrt[K] r)/Sqrt[DM]])/K]}}

C1 = CP[[1, 1, 2, 2]]
(-UMCapdc + UMCapdc BesselJ[0, (i Sqrt[K] r)/Sqrt[DM]])/K

r = rc; Cc = C1
Clear[r]
(-UMCapdc + UMCapdc BesselJ[0, (i Sqrt[K] rc)/Sqrt[DM]])/K
```

B.2 Coding of Concentration in outer region ($\bar{C}_2$)

```
File  Edit  Insert  Format  Cell  Graphics  Evaluation  Palettes  Window  Help

C2

In[82]:= Clear[r, rc]; Cc

Out[82]= (-UMCapdc + UMCapdc BesselJ[0, (i Sqrt[K] rc)/Sqrt[DM]])/K

In[83]:= Clear[z, zc, r, rc]; z = r / Rz; Clear[CP2];
UPCap = UN (1 - 2 z^2 - 64/21 zc^(1/2) + 8/3 zc + 1/21 zc^4 + 16/3 zc^(1/2) z^(3/2) - 4 zc z);
CP2 = DSolve[{DM/r C''[r] + C'[r] - UPCapdc - K C[r] == 0, C'[Rz] == 0, C[rc] == Cc}, C, r]

C2 = CP2[[1, 1, 2, 2]]
```

B.3 Coding of I_1

```
File  Edit  Insert  Format  Cell  Graphics  Evaluation  Palettes  Window  Help

I1

Clear[Bzc, r, rc, I1A]; I1A[r_] = UMCap C1 r;
Simpson38[a0_, b0_, m0_] := Block[{as = N[a0], bs = N[b0], hs, m = m0, r},
    hs = (bs - as)/(3 m); r_s_ = as + s hs;
    Return[(3 hs)/8 (Sum[(I1A[r_{3 s-3}] + 3 I1A[r_{3 s-2}] + 3 I1A[r_{3 s-1}] + I1A[r_{3 s}]), {s, 1, m}])];];
I1A = Simpson38[0, rc, 3]
clear[I1]; I1 = Collect[I1A, {Bzc, Rz}]
```

B.4 Coding of I_2

```
File  Edit  Insert  Format  Cell  Graphics  Evaluation  Palettes  Window  Help

I2

ClearAll[I2A, Q1, Q2, Q3]; I2A[r_] = UPCap C2 r R0^2;
" Integrate[uPCap C2 r R0 R0, {r, rp R0, Rz R0}] = Integrate[UPCap C2 r R0^2, {r, rp R0, Rz R0}]";
Simpson38[a0_, b0_, m0_] := Block[{as = N[a0], bs = N[b0], hs, m = m0, r},
    hs = (bs - as)/(3 m); r_s_ = as + s hs;
    Return[(3 hs)/8 (Sum[(I2A[r_{3 s-3}] + 3 I2A[r_{3 s-2}] + 3 I2A[r_{3 s-1}] + I2A[r_{3 s}]), {s, 1, m}])];];
I2A = Simpson38[rc R0, Rz R0, 3];
$Aborted
Clear[I2]; I2 = Collect[I2A, {dc, K, UN, Rz}]
```

B.5 Coding of *E*

```
File  Edit  Insert  Format  Cell  Graphics  Evaluation  Palettes  Window  Help

Clear[Q1]; Q1 = −DM dc + 2/R0^2 (I1 + I2)

Clear[Q2]; Q2 = Q1/(−DM dc)

Clear[Q3]; Q3 = Q2 − 1

Clear[Ezc]; Ezc = (Q3 48 DM^2)/(Rz^2 UN^2) // Re
```

B.6 Coding of Normalized Velocity

```
File  Edit  Insert  Format  Cell  Graphics  Evaluation  Palettes  Window  Help

Velocity of Casson fluid ( Normalized velocity )

Larger the size label

ClearAll[n, rc, ur, um, u, r];
rc = 0.1;
ur = (−1)/(4 μ) dp (Rz^2 − r^2 − 8/3 rc^(1/2) (Rz^(3/2) − r^(3/2)) + 2 rc (Rz − r));
r = rc; Rz = 0.8 ;
urc = ur;
Clear[r];
um = (−Rz^2)/(8 μ) dp (1 − 16/7 rc^(1/2)/Sqrt[Rz] + 4/3 rc/Rz − 1/21 rc^4/Rz^4);
u1 = Re[urc/um];
u2 = Re[ur/um];


Vel[r_] := Which[0 ≤ r ≤ rc, u1, rc ≤ r ≤ 1, u2];
Print[TableForm[ParallelTable[{r, Vel[r]}, {r, 0, 1, 0.1}], TableHeadings → {None, {"r", "Velocity"}}]]
ParametricPlot[{{Vel[r], r}, {Vel[r], −r}}, {r, −1, 1}, FrameLabel → {"Normalized Velocity, u/um", "Radius, r"},
  Frame → True, ImageSize → 80 × 5, LabelStyle → Directive[20], PlotStyle → {Black}, AspectRatio → 0.9, PlotRange → {{0, 2}, {−1, 1}}]
```

B7. Coding of Distribution Concentration

File Edit Insert Format Cell Graphics Evaluation Palettes Window Help

```
ClearAll[μ, Rz, dp, Bzc, a, Z, rc, r, DM, del, l0];
r = rc; Cp = C1
Clear[r]

μ = 1;
dp = 1;
dc = 1;
DM = 1;
l0 = 2;
a = 0.7;
del = 0.1;
zc = rc / Rz;
z = r / Rz;
d = 3;
Z = 4;
Rz = a - del (1 + Cos[(2 Pi / l0) (Z - d - l0/2)]);
UN = (-Rz^2 / (8 μ)) dp;
Bzc = 1 - (64/21) (zc)^(1/2) + (8/3) (zc) - (2/3) (zc)^2 + (1/21) (zc)^4;


Conc[r_] := Which[0 ≤ r ≤ rc, Re[C1], rc ≤ r ≤ 1, Re[C2]];

Do[Print["del = ", del]; Print["rc = ", rc]; Print["Rz = ", Rz]; Print["K = ", K];
  Print[TableForm[ParallelTable[{r, Conc[r]}, {r, 0, 1, 0.1}], TableHeadings → {None, {"r", "Concentration"}}]], {rc, 0.1, 0.3, 0.1}, {K, 0.1, 0.1, 0.1}]
ConcFig = ParametricPlot[{{Conc[r], r}, {Conc[r], -r}}, {r, -1, 1}, FrameLabel → {"C", "r"}, Frame → True, ImageSize → 80 × 5,
   LabelStyle → Directive[20], PlotStyle → {Black}, AspectRatio → 0.9, PlotRange → Automatic]
```

B.8 Coding of Effective Axial Diffusion

File Edit Insert Format Cell Graphics Evaluation Palettes Window Help

Deff/Dm

Effective axial diffusivity with stenosis

```
Clear[Azc, Bzc, Ezc, RED, UN, rc, Rz, zc, r, dc, R0, a, DM, α, K, Pe];
K = 1; Rz = 0.6; dc = 1; UN = 1; DM = 1; a = 1; R0 = 1; zc = rc / Rz; r = rc;
Bzc = 1 - (64/21) (zc)^(1/2) + (8/3) (zc) - (2/3) (zc)^2 + (1/21) (zc)^4;
Azc = 1 - (16/7) zc^(1/2) + (4/3) zc - (1/21) zc^4;

Ezc = (Q3 48 DM^2) / (Rz^2 UN^2) // Re;
DeffDm = 1 + (Pe^2/48) (Ezc / Azc^2) // Re;

Do[Print[" "]; Print["Pe = ", Pe];
  Print[TableForm[Table[{rc, zc, Re[Ezc], Azc, DeffDm}, {rc, 10^-4, 0.21, 0.01}],
     TableHeadings → {None, {"rc", "zc", "Ezc", "Azc", "Deff/Dm"}}, TableSpacing → {3, 4}]], {Pe, 0, 20, 4}]
```

B.8 Coding of Relative Axal Diffusion

File Edit Insert Format Cell Graphics Evaluation Palettes Window Help

```
Clear[Azc, Bzc, RED, UN, rc, Rz, zc, r, dc, R0, a, DM, α, K, Pe];
K = 0.1; dc = 1; UN = 1; DM = 1; a = 1; R0 = 1; zc = rc / Rz ; r = rc ;
Bzc = 1 - 64/21 (zc)^(1/2) + 8/3 (zc) - 2/3 (zc)^2 + 1/21 (zc)^4;
Azc = 1 - 16/7 zc^(1/2) + 4/3 zc - 1/21 zc^4;

Ezc = (Q3 48 DM^2)/(Rz^2 UN^2) // Re;
Rad = (Ezc/Azc^2) // Re;


Do[Print[" "];["Rz = ", Rz];
  Print[TableForm[Table[{rc, zc, Re[Ezc], Azc, Rad}, {rc, 10^-4, 0.21, 0.01}],
      TableHeadings → {None, {"rc", "zc", "Ezc", "Azc", "Relative axial diffusivity"}}, TableSpacing → {3, 4}]], {Rz, 0.6, 1, 0.1}]
```