LEAST-SQUARES FINITE ELEMENT METHOD FOR SOLVING STOKES EQUATION WITH A POINT SOURCE MAGNETIC FIELD

ALIA RAFIZA BINTI CHE AYOB

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DEDICATION

To my beloved ma and abah,

Wan Jamilah Binti Wan Ali,

Che Ayob Bin Hussin,

My husband,

Muhammad Faizul Bin Gulam Shariff,

And

My family.

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ABSTRACT

The formulation and numerical computation of the two-dimensional Stokes flow under the effect of a point source magnetic field are presented in this study. Stokes flow is also known as low Reynolds number, creeping flow, or non-inertial. Least-squares finite element method (LSFEM) is successfully employed for the discretization of the Stokes equation. LSFEM has several advantages in terms of theory and computing, and it can create a symmetric and positive-definite algebraic system of equations that can be solved quickly and robustly using iterative approaches. However, LSFEM is having an issue where the low order nodal expansions tend to lock. Thus, the present study proposed the discretization of the problem domain using higher-order elements. The source codes for the Stokes equation with and without the point source magnetic field effect have been developed and verified against the existing benchmark solutions. The verification achieved an excellent agreement. The solution of the Stokes flow in a lid-driven cavity and a straight rectangular channel subjected to the point source magnetic field are conducted. The results concerning velocity contour and streamlines pattern are analysed. Firstly, the streamlines pattern in the lid-driven cavity problem shows the development of a vortex at the bottom-left corner cavity. The new vortex appeared as the secondary flow in cavity. As the magnetic number grows, the primary flow separates from the secondary flow. Secondly, when the straight rectangular channel problem was solved, a single vortex emerged at the channel's lower wall which is close to the point of the magnetic source. As the magnetic number increased, a new vortex appeared at the channel's upper wall. This shows that the point source magnetic field has a substantial impact on Stokes flow, as shown by the numerical simulation findings. Based on the current results, it can be concluded that the LSFEM can be used to solve Stokes flow problems with the effect of the point source magnetic field.

ABSTRAK

Perumusan dan pengiraan berangka untuk aliran Stokes dua dimensi di bawah pengaruh titik sumber medan magnet dibentangkan dalam kajian ini. Aliran Stokes juga dikenali sebagai nombor Reynolds rendah, aliran menjalar, atau tidak inersia. Kaedah unsur terhingga kuasa dua terkecil (LSFEM) berjaya digunakan untuk pendiskretan persamaan Stokes. LSFEM mempunyai beberapa kelebihan dari sudut teori dan pengkomputeran, dan LSFEM dapat menghasilkan sistem persamaan algebra yang simetrik dan tentu positif yang dapat diselesaikan dengan cepat dan teguh menggunakan kaedah pelelaran. Namun, LSFEM menghadapi masalah di mana nod tambah peringkat rendah cenderung untuk terkunci. Oleh itu, kajian ini mencadangkan pendiskretan masalah domain menggunakan elemen peringkat tinggi. Kod program untuk persamaan Stokes dengan dan tanpa kesan titik sumber medan magnet telah dibangunkan dan disahkan dengan penyelesaian tanda aras yang sedia ada. Pengesahan ini mencapai persetujuan yang sangat baik. Penyelesaian aliran Stokes di dalam rongga yang tertutup dan saluran segi empat tepat lurus yang tertakluk kepada titik sumber medan magnet telah dijalankan. Hasil yang diperoleh untuk kontur halaju dan corak garis arus dianalisis. Pertama, hasil dari corak garis arus dalam masalah rongga yang tertutup menunjukkan bahawa vorteks berkembang berhampiran bawah sudut kiri rongga. Vorteks baharu itu muncul sebagai aliran kedua dalam rongga. Apabila nilai magnet bertambah, aliran utama terpisah dengan aliran kedua. Kedua, apabila masalah saluran segiempat tepat lurus diselesaikan, vorteks tunggal terbentuk di dinding bawah saluran yang berhampiran dengan titik sumber magnet. Apabila nilai magnet meningkat, vorteks baharu terbentuk di dinding atas saluran. Ini menunjukkan bahawa titik sumber medan magnet mempengaruhi aliran Stokes secara signifikan. Berdasarkan hasil dari kajian ini, boleh disimpulkan bahawa LSFEM boleh digunakan untuk menyelesaikan masalah aliran Stokes dengan kesan titik sumber medan magnet.

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LIST OF ABBREVIATIONS

FEM	-	Finite element method
LSFEM	-	Least-squares finite element method
FHD	-	Ferrohydrodynamics
MHD	-	Magnetohydrodynamics
BFD	-	Biomagnetic fluid dynamics
FDM	-	Finite difference method
FVM	-	Finite volume method
BEM	-	Boundary element method
PDE	-	Partial diefferential equation
DRBEM	-	Dual reciprocity boundary element method
MQ	-	Multiquadric method
SBM	-	Singular boundary method
MLS	-	Moving least-squares
WG	-	Weak Galerkin
RBF	-	Radial basis function
RBF-DQ	-	Radial basis function-based differential quadrature
MAPS	-	Method of Approximate Particular Solutions
CSCM	-	Chebyshev spectral collocation method
LBB	-	Ladyshenskaya-Babuška-Brezzi
CFD	-	Computational fluid dynamics

LIST OF SYMBOLS

ρ	-	Density
$\bar{\mu}$	-	Dynamic viscosity
$\bar{\mu}_0$	-	Magnetic permeability of vacuum
χ	-	Magnetic susceptibility
γ	-	Magnetic field strength
$\bar{u}$	-	Velocity vector
$\bar{u}_r$	-	Maximum velocity
$\bar{u}(\bar{y})$	-	Parabolic velocity profile
p	-	Pressure
$\bar{M}$	-	Magnitude of magnetization
$\bar{H}$	-	Magnitude of magnetic field intensity
$\bar{h}$	-	Height of cavity
$\bar{L}$	-	length of the cavity
Mn	-	Magnetic number
(ξ, η)	-	Local coordinate
N_i	-	Serendipity interpolation functions
Ω^e	-	Typical element
I^e	-	Least-squares functional over typical element
$[J]$	-	Jacobian matrix
w_i	-	Gauss weight
K	-	Stiffness matrix
U	-	Vector of the degree of freedom
F	-	Force vector

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CHAPTER 1

INTRODUCTION

1.1 Problem Background

A boundary value problem is a system of differential equations to be solved in a domain subject to the boundary conditions on the unknown function specified at two or more values of the independent variables (Ricardo, 2020). Fluid dynamics, magnetohydrodynamics, ferrohydrodynamics are the problems that can be modelled in terms of boundary value problems.

Fluid dynamics is the study of fluids in motion where the pressure forces are considered. Fluid is a substance that continually flows when influenced by shear stress (force per unit area) of any magnitude such as liquid, gases, and plasmas (Munson et al., 2013). The application of fluid dynamics can be found in almost every area in our daily lives, such as the circulation of blood, hurricanes, airflow over planes and the flow of water in rivers or pipes. Fluid motion is usually influenced by the magnetic field and electromagnetic field and the interaction of the fluid with the magnetic field is made use of in MRI medical exams, nuclear fusion, transformer cooling, and more (Şenel, 2017).

Ferrohydrodynamics (FHD) is a branch of fluid mechanics that study how magnetic polarisation affects fluid velocity. FHD deals with no electric current and considered that the flow is affected by the magnetization property of fluid in the magnetic field. While, magnetohydrodynamics (MHD) deals with the motion of an electrically conducting fluid in the presence of a magnetic field. The motion of conducting material across the magnetic lines of force creates potential differences which, in general, cause electric currents to flow. The body force in FHD is due to polarisation force, which in turn necessitates material magnetization when there are magnetic field gradients. In MHD, the flow of electric current across a magnetic field is associated with a body force, which called Lorentz force, that influences the fluid (Rosensweig, 2013; Roberts, 1967).

The fundamental governing equations of fluid dynamics consist of continuity, momentum, and energy equation which are expressed in terms of partial differential equations (PDE). These governing equations are upon three fundamental principles which are based on: (1) mass is conserved; (2) Newton's second law (F = ma); and (3) energy is conserved (Anderson and Wendt, 1995). These governing equations also required a numerical approach to obtain the solution since PDE cannot be solved analytically. Different kinds of discretization can be used to formulate a PDE approximation. Methods of discretization approximate the PDE utilizing numerical model equations, which may be solved numerically.

Some essential characteristics of Stokes flow, such as negligibility of inertial forces, reversibility, and the minimal energy dissipation theorem. The nonlinear Navier-Stokes equation simplify to linear Stokes equation. The Navier-Stokes equation for an unsteady and viscous incompressible fluid are shown as

$$\bar{\rho}\frac{\partial \bar{u}}{\partial \bar{t}} + \bar{\rho}(\bar{u}\cdot\nabla)\bar{u} = \bar{\mu}\nabla^2\bar{u} - \nabla\bar{p} + \bar{F}\,, \tag{1.1}$$

$$\nabla\cdot\bar{u} = 0\,, \tag{1.2}$$

where $\bar{\rho}$ is the fluid density, $\bar{u}$ is the fluid velocity field, $\bar{\mu}$ is the dynamic viscosity, $\bar{p}$ is the pressure of fluid and $\bar{F}$ is the external forces. Equation 1.1 has inertial forces on the left and viscous and pressure forces on the right, as well as any external body forces acting on the fluid element. This equation is predicated on the fluid elements incompressibility, which allows it to take on a simple form. It is necessary to utilise non-dimensionalisation to accurately depict the relative magnitude of forces.

According to the Navier–Stokes equation, a dimensionless equation version is resulting as

$$Re\left(\frac{\partial u}{\partial t} + u\cdot\nabla u\right) = \nabla^2 u - \nabla p + F\,, \tag{1.3}$$

where Re is dimensionless Reynolds number, which defined as the ratio of inertial and viscous forces in a fluid. The left-hand side Equation (1.3) is the inertial force and the right-hand side Equation (1.3) is the viscous, pressure and body forces.

When it comes to forces, the Stokes approximation occurs when viscous and pressure forces completely outweigh inertial forces. The Reynolds number can be used to determine whether or not Stokes approximation can be used to model fluids. As the ratio approaches 0, the Stokes approximation holds perfectly. As a result, Stokes flows are usually referred to as low Reynolds number, non-inertial, or viscous flows when discussing fluid dynamics.

When Re is approach to 0 in Equation (1.3), the dimensionless steady Stokes equation is obtained. While, in dimensional form, neglected external forces, sources, or sinks, the equations become

$$\bar{\mu}\nabla^2\bar{u} - \nabla\bar{p} = 0\,, \tag{1.4}$$

$$\nabla \cdot \bar{u} = 0\,. \tag{1.5}$$

Equation (1.4) depicts the balance of forces in a non-accelerating fluid, whereas Equation (1.5) depicts the mass conservation in incompressible fluids. For stiff walls and particle surfaces, the Stokes equation must be coupled with boundary conditions that are relevant to the physical environment. A surface, S moves with local velocity, $\bar{w}$. The fluid velocity, $\bar{u}$ has a no-slip boundary condition if on S has

$$\bar{u}(\bar{r}) = \bar{w}(\bar{r}) \qquad \text{for } \bar{r} \in S\,. \tag{1.6}$$

It is necessary to focus on no-slip boundary conditions when taking input from inertial forces, reversibility, and minimal energy dissipation. Since equations (1.4) and (1.5) are linear, classes of solutions may be created, such as for flow around a hard-sphere (Trombley and Ekiel-Jeewska, 2019).

PDE numerical problems are often solved using methods such as the finite difference method (FDM), the finite volume method (FVM), the finite element method (FEM), and the boundary element method (BEM) (Venkateshan and Swaminathan, 2013). Among these, the FEM is used to compute such approximations since FEM has a benefit as it offers great freedom in the selection of discretization, both in the elements and in the basis function. Many possible finite element method formulations can be used for discretization such as Galerkin, weak-form Galerkin, least-squares, subdomain, and so on. Least-squares finite element method (LSFEM) has advantages over the Galerkin and weak-form Galerkin in several theoretical and computational for viscous incompressible flows. Equal-order interpolations may be accommodated using LSFEM based on velocity-vorticity formulation and the resulting matrices are always symmetric, positive, and definite. Discrete systems of equations may be solved using robust iterative techniques due to symmetric positive-definiteness (Reddy and Gartling, 2010).

The present study is interested in solving the Stokes equation subjected to the point source magnetic field problem numerically by using LSFEM.

1.2 Problem Statement

The numerical approach method is frequently utilised to solve problems in a wide range of engineering and applied scientific disciplines. When the problems in real environments get more difficult and complex, new numerical techniques are needed. To relate with the real environmental problems, it is a must to also considered other effects and this usually results in coupled partial differential equations. Magnetic field effect has become one of the main interests in applications of medical sciences and bioengineering. Magnetic devices for cell separation, targeted medication delivery, magnetic cancer tumour therapy, bleeding reduction during surgery, and magnetic tracers are just a few of the new uses for magnets being developed. FEM is one numerical method that possible to obtain the solution of the problem that involves the magnetic field effect. Besides, FEM is a well-established and dependable numerical method for dealing with complex geometries, and it often achieves excellent accuracy with a coarser mesh than FDM and FVM. However, finite element formulation poses a stability issue in the presence of highly nonlinear body force. Based on the study by Abdullah et al. (2020), they are facing numerical instability when using the Galerkin weighted residual finite element method due to extremely steep magnetic field gradient. In order to mitigate this issue, they introduced a stabilization term in their FEM formulation. Therefore, it presents a new challenge in finding other formulations that can be employed to obtain the solution. Among available formulations, LSFEM will be used since the numeric developments of this method has advantages that can always obtain symmetrical and defined positive algebraic system. The least-squares formulation is quite simple to use in the discretization of governing equations since these equations can be transformed to an equivalent first-order system. However, previous research emphasizes that the LSFEM has an issue where the low order nodal expansions tend to lock. Locking occurs in lower order elements because the elements kinematics are insufficient to represent the correct solution. It means the effect of a reduced rate of convergence in dependence of a parameter. Thus, the present study also needs to find a solution to solve this kind of issue. This is yet another attempt to solve the fluid flow with the effect of a point source magnetic field using such a formulation.

1.3 Research Objectives

The objectives of the research are:

(a) to discretize the Stokes equation with point source magnetic field and the boundary conditions using least-squares finite element method (LSFEM) and to construct mesh for the case study problem using Gmsh software

(b) to develop MATLAB code based on the numerical discretization that has been performed by using LSFEM model

(c) to validate numerical formulation with benchmark problem

(d) to analyse the effect of magnetic field in the lid-driven cavity and straight rectangular channel.

1.4 Scope of Study

This research focused on numerical modelling of Stokes equation under a point source magnetic field. It is numerically done using the finite element method with least-squares formulations. In a lid-driven cavity and a straight rectangular channel problem, fluid flow is exposed to an external point source magnetic field. To introduce magnetic force into a flow, a spatially varying magnetic field is utilised. The flow is presumed to be two-dimensional, steady, Newtonian, electrically non-conducting, laminar and incompressible. The dimensional Navier-Stokes equation of the fluid flow under the effect of point source magnetic field are similar to those derived by Rosenweig (2013) and Senel and Tezer-Sezgin, (2016). While, the dimensionless Stokes equation for the problem study considered is the same as used by Senel and Tezer-Sezgin, (2016). The Stokes flow problem in the current study is extended from the study by Young and Yang (1996) with the addition of the point source magnetic field by referred to the study by Senel and Tezer-Sezgin (2016).

The solution in this study is obtained using the least-squares finite element method (LSFEM). Least-squares formulations is used to discretize the governing equations and the boundary conditions. Two problem domains which are a lid-driven cavity and a straight rectangular channel are discretized using rectangular element. The meshing of the problem is constructed by using Gmsh software. Gmsh is an external mesh generator which can easily create geometries and meshes and it can export the mesh directly to MATLAB software by using post-processor. Then, MATLAB source code is developed in order to obtain the solution. Velocity contours and streamlines are used to represent the numerical results for various magnetic number.

1.5 Significance of Research

This study concerns the finite element formulations for the coupled problem of the fluid dynamics with influences of the magnetic field. The significance of this study would be on the first application of the LSFEM in the study of Stokes flow under the effect of a point source magnetic field. With the approach of this study, it is hoped that an understanding of the behaviour of the flow, when subjected to an external point source magnetic field, can be obtained. Effects such as the disturbance in the velocity of the flow and the formation of the vortex can be used for practical purposes in medical and bioengineering. Furthermore, this study will serve as a springboard for future research into coupled fluid dynamics problems in finite element modelling.

1.6 Overview of Thesis

The present chapter gives a brief introduction to fluid dynamics and the numerical approach. There is a problem statement, followed by a research aim and objectives, that explains why the study is needed. The importance and scope of the study are also discussed at length at the end of this chapter.

The rest of the thesis is divided into five chapters. A review of the literature is given in Chapter 2. The chapter begins with research on Stokes flow. An overview of coupled fluid dynamics and magnetic field effect problem is presented afterward emphasizing the FHD and MHD principles. Several numerical methods typically used for the solution of Stokes equation are presented with their advantages and disadvantages highlighted. Then, the LSFEM for the solution of the governing equations is discussed thoroughly.

In Chapter 3, the Navier-Stokes equation under the magnetic field effect is presented in dimensional form. Then, the nondimensional variables are used to transform these equations into dimensionless form. After applied the conditions for the Stokes flow case, the Stokes equation subjected to the magnetic field are presented. Modelling of Stokes flow under magnetic field effect problem by the LSFEM is conducted. Then, the dimensionless Stokes equation are discretized using the least-squares formulation.

In Chapter 4, the finite element method numerical scheme is presented briefly. Mesh for the discretization of the domain is constructed by using Gmsh and the usage of Gmsh is demonstrated step by step. The source code that has been developed is validated with the previous research and COMSOL Multiphysics 5.2 software.

Chapter 5 focused on the results of the Stokes problem. Two problems in which the Stokes flow under magnetic field effect in the lid-driven cavity and straight rectangular channel are considered. This chapter presents the results on the effect of the applied magnetic field in both problems. The results concerning velocity and streamline are observed and discussed.

Overall results from this study are provided in Chapter 6 of this thesis as a conclusion. The simulation's results are in, and many takeaways for further research are offered.

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3. Ayob, A. R. C., Ismail, Z., and Kasiman, E. H. (2022). Least-Squares Finite Element Method for Solving Stokes Flow under Point Source Magnetic Field. *Symmetry, 14*(3), 514.

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