THE CONJUGACY CLASSES AND RELATED GRAPHS OF SOME THREE-GENERATOR p-GROUPS

ALIA HUSNA BINTI MOHD NOOR

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DEDICATION

This thesis is dedicated to my beloved parents, who encouraged me to further my studies, my husband, who always supported me in what i do and my siblings who always pray the best for me. It is also dedicated to my supervisor, Prof Dr Nor Haniza Sarmin who never failed to guide me.

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May Allah grant all the kindness to all of them.

ABSTRACT

A conjugacy class is a set of elements in the group under the conjugation action. Meanwhile, a graph consists of points called vertices, and links which are called edges, is indicated by line segments or curves joining certain pairs of vertices. In the first part of this research, the conjugacy class of each type of groups in the classification is computed based on the definition of conjugacy class itself. Previous researchers have found the lower and upper bound of conjugacy classes. Compared to those researches, this research move a step further where the exact number of conjugacy classes of groups are computed. These results are then applied into finding its conjugacy class graph. In the second part of this research, the conjugacy classes of the groups satisfying the set omega is determined. A set omega is defined as the set of all ordered pairs of commuting elements in the classification. In this research, the classification of 3-generator p-groups of certain order is used. The conjugacy classes is computed by using several steps. Firstly, the elements of order p is identified, followed by the determination of the number of elements of ordered pairs in the set omega. The number of conjugacy classes is then obtained and expressed in a general form. The obtained results are later associated to two types of graph, namely the orbit graph and the generalized conjugacy class graph. For the orbit graph, two vertices are adjacent if they are conjugated to each other. Meanwhile, for the generalized conjugacy class graph, two vertices are adjacent to each other if their cardinalities are not coprime. The graphs turned out to be a complete graph or a union of complete graphs with p vertices. In the last part of this research, two new graphs of 3-generator 3-groups called the generalized commuting conjugacy class graph and the generalized non-commuting conjugacy class graph are introduced. The generalized commuting conjugacy class graph is a graph whose the vertices are the elements of the conjugacy classes in the form of ordered pairs. Two vertices are adjacent if they are commuted. Since the generalized non-commuting conjugacy class graph is a complement of the generalized commuting conjugacy class graph, thus, the edges are connected if their vertices are not commuted.

ABSTRAK

Kelas kekonjugatan adalah suatu set unsur di dalam suatu kumpulan di bawah tindakan kekonjugatan. Sementara itu, graf adalah terdiri daripada titik-titik yang dipanggil bucu-bucu, dan sambungan-sambungan yang dipanggil sisi-sisi, diwakilkan oleh segmen garis atau lengkung yang menyambungkan pasangan bucu-bucu tertentu. Pada bahagian pertama kajian ini, kelas kekonjugatan bagi setiap jenis kumpulan dalam klasifikasi dikira berdasarkan definisi kelas kekonjugatan itu sendiri. Penyelidik-penyelidik sebelum ini telah mencari batas bawah dan atas bagi kelas kekonjugatan. Berbanding kajian-kajian tersebut, kajian ini melangkah lebih jauh yang mana bilangan kelas kekonjugatan yang tepat telah dikira. Hasil ini kemudiannya digunakan untuk mencari graf kelas kekonjugatannya. Pada bahagian kedua kajian ini, kelas kekonjugatan bagi setiap kumpulan yang menepati ciri-ciri dalam set omega ditentukan. Suatu set omega ditakrifkan sebagai set pasangan tertib yang saling tukar tertib dalam klasifikasi tersebut. Kelas kekonjugatan dikira dengan menggunakan beberapa langkah. Pertama, unsur-unsur peringkat p dikenalpasti, diikuti oleh penentuan set pasangan tertib dalam set omega. Bilangan kelas kekonjugatan kemudiannya diperoleh dan dinyatakan dalam bentuk umum. Hasil yang didapati kemudiannya dikaitkan dengan dua jenis graf, iaitu graf orbit dan graf kelas kekonjugatan teritlak. Bagi graf orbit, dua bucu adalah bersebelahan jika mereka saling berkonjugat di antara satu sama lain. Sementara itu, bagi graf kelas kekonjugatan teritlak, dua bucu adalah bersebelahan jika kardinaliti bucu-bucu tersebut bukan perdana relatif. Graf tersebut ternyata adalah graf lengkap atau kesatuan graf lengkap dengan bilangan bucu p. Pada bahagian terakhir kajian ini, dua graf baharu kumpulan-3 berpenjana-3 dikenali sebagai graf kelas kekonjugatan teritlak kalis tukar tertib dan graf kelas kekonjugatan teritlak bukan kalis tukar tertib diperkenalkan. Graf kelas kekonjugatan teritlak kalis tukar tertib adalah suatu graf di mana bucunya terdiri daripada unsur-unsur dalam kelas kekonjugatan dalam bentuk pasangan tertib. Dua bucu adalah bersebelahan jika bucu-bucu tersebut kalis tukar tertib. Memandangkan graf kelas kekonjugatan teritlak bukan kalis tukar tertib adalah pelengkap kepada graf kelas kekonjugatan teritlak kalis tukar tertib, maka, sisinya terbentuk jika bucu-bucu tersebut tidak kalis tukar tertib.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

LIST OF SYMBOLS

A_n	-	Alternating group
$Z(G)$	-	Center of a group G
A	-	Commuting elements in the set Ω
$[x,y] = xyx^{-1}y^{-1}$	-	Commutator of x and y
K_n	-	Complete graph with n vertices
$cl(a)$	-	Conjugacy Class of a
Γ_G^{cl}	-	Conjugacy Class Graph of G
D_n	-	Dihedral group of order $2n$
$\bar{K}_n$	-	Empty graph with n vertices
$\Gamma_G^{\Omega_{Ccl}}$	-	Generalized Commuting Conjugacy Class Graph of G
$\Gamma_G^{\Omega_{cl}}$	-	Generalized Conjugacy Class Graph of G
$\Gamma_G^{\Omega_{nCcl}}$	-	Generalized Non-Commuting Conjugacy Class Graph of G
Γ_G	-	A graph of a group G
K_0	-	A graph with no vertices
$K(G)$	-	Number of conjugacy classes in G
$K(\Omega)$	-	Number of conjugacy classes in the set Ω
Γ_G^{Ω}	-	Orbit Graph of G
Ω	-	A set an ordered pair of elements in the group with certain rules
$E(\Gamma)$	-	A set of edges
$V(\Gamma)$	-	A set of vertices
S_n	-	Symmetric group of order n
$V(\Gamma_G)$	-	Vertices of a graph of group G

CHAPTER 1

INTRODUCTION

1.1 Introduction

Conjugation is one of the actions on sets besides many other group actions, such as transitive, faithful, free, regular, n-transitive, sharply n-transitive, primitive and locally free. Let G be a group and X is a set in G. The group action of G on X is one of the ways how the elements in group G are moved around on elements in X. When a group G acts on X, each element $x \in X$ has an orbit. The orbit is defined as $Orb(x) = \{gx|g \in G, x \in X\}$. Note that the action focused in this research is conjugation action. Hence, the orbit now is defined as $gx = gxg^{-1}$ for the conjugation action of G on X where $g \in G$ and $x \in X$. The orbit of the elements of this conjugation action is known as conjugacy class. Some researchers have studied on conjugation in order to compute the conjugacy class and related it to graph theory.

A conjugacy class is a set of elements in the group under the conjugation action. Besides, a conjugacy class is an equivalence class under the equivalence relation of being conjugate. The operation of conjugacy is defined as follows; Let a, b be the elements in a group G. Then a and b are conjugate if for some $g \in G$, $gag^{-1} = b$. The conjugacy class of a is written as $cl(a) = \{x^{-1}ax|x \in G\}$, while the conjugacy class of a group G is denoted by $cl(G)$.

In graph theory, a graph consists of points which are called vertices, and connections which are called edges, which are indicated by line segments or curves joining certain pairs of vertices. In this research, the results on conjugacy classes of 3-generator groups of order p^4 will be applied into graph theory to obtain the conjugacy class graph, generalized conjugacy class graph, orbit graph, generalized commuting conjugacy class graph and generalized non-commuting conjugacy class graph.

1.2 Background of the Research

The study on conjugacy class has started many years ago by Erdos and Turan [1] who worked on some problems of a statistical investigation of S_n, the symmetric group of n letters. In their paper, one of the problems considered is the number of conjugacy classes of S_n. Over the next years, the study on conjugacy classes has been grown widely specifically on the lower bound of the conjugacy classes. In 2014, Liu and Song [2] made an improvement on the lower bound found by He and Shi [3] for the largest conjugacy class length of a finite group. In their paper, they investigated the largest conjugacy class length of almost simple groups.

The findings of the number of conjugacy classes have attracted many researchers to work on, involving different types of group. In 2015, Naphtali *et al.* [4] have showed the counting of the conjugacy classes of finite groups by using the centralizer. In their paper, they used the concept of class equation as a tool to count the conjugacy classes for finite non-abelian groups of prime power order. In class equation, the order of the group is the summation of the elements from each conjugacy classes and the center. Besides, they also rely on the fact that the number of conjugacy classes for the non-abelian case is less than the order of the group. Additionally, they defined the conjugacy class as an equivalence relation where the equivalence classes are the conjugacy classes. From their finding, they proved that the upper bound of the conjugacy classes of finite non-abelian group of order p^w is $\frac{1}{4}(2p^w + p^r)$ where $3 \leq w \leq 6$ and p^r is the order of the centralizer of an element x, and w and r are considered as positive integers such that r is less that w. It can be seen that, some study on the number of conjugacy classes from previous research have restricted to the lower bound and upper bound of conjugacy classes.

In 2012, Ahmad *et al.* [5] found the formula for the number of the conjugacy classes for 2-generator p-groups of nilpotency class 2 with order p^n and has derived subgroup of order p^γ in term of n and γ i.e $K(G) = p^{n-\gamma}(1 + p^{-1} - p^{-(\gamma+1)})$. The group representation which is used in this paper is given in the following theorem.

Theorem 1.1 Let p be a prime and $n > 2$ a positive integer. Every 2-generated p-group of class 2 exactly correspond to an ordered 5-tuple of integer $(\alpha, \beta, \gamma, \rho, \sigma)$ such that

1. $\alpha \geq \beta \geq \gamma \geq 1$,

2. $\alpha + \beta + \gamma = n$,

3. $0 \leq \rho \leq \gamma$ and $0 \leq \sigma \leq \gamma$

where $(\alpha, \beta, \gamma : \rho, \sigma)$ corresponds to the group presented by

$$G : \langle a, b | [a, b]^{p^\gamma} = [a, b, a] = [a, b, b] = 1, a^{p^\alpha} = [a, b]^{p^\rho}, b^{p^\beta} = [a, b]^{p^\alpha} \rangle.$$

Based on this group representation, they find the number of conjugacy classes by applying the Euler's Totient function. The number of conjugacy classes of this 2-generator p-group is written as follows.

Theorem 1.2 Let G be a 2-generator p-group of nilpotency class 2. If G has order p^n and has derived subgroup of order p^γ, then G has $p^{n-\gamma}(1 + p^{-1} - p^{-(\gamma+1)})$ conjugacy classes.

By using the Euler's Totient Function, they started the proof with the number of conjugacy classes of G is $p^{\alpha+\beta-\gamma} + \sum\limits_{\delta=1}^{\gamma} C_\delta(G)$ where $1 \leq \delta \leq \gamma$. The $p^{\alpha+\beta-\gamma}$ is actually the center of the group. Meanwhile, $\sum\limits_{\delta=1}^{\gamma} C_\delta(G)$ is the summation of the number of the non-central conjugacy classes with each possible order. Hence, they proved that the total number of conjugacy classes is the summation of the number of conjugacy classes of each order i.e $p^{n-\gamma}(1 + p^{-1} - p^{-(\gamma+1)})$. The truth is the concept of class equation is also applied in this proving.

Besides, the number of conjugacy classes has been found for symmetric group and dihedral group in [6] and alternating group in [7]. Various number of researches on conjugacy classes have been studied such as in [9 - 13].

In this research, the methods in the computation of conjugacy classes are connected with the previous researchers from [4] and [5] where the formula of the class equation and the basic concepts of Euler phi function are applied. In addition, some basic knowledges on the order of the elements in a group are also referred especially in the computation of conjugacy classes in Chapter 4. However, instead of using generator a, b with parameters $(\alpha, \beta, \gamma, \rho, \sigma)$ by Ahmad, the variables x, y and z without parameter are used as generators suitable with the group presentation given by [15] since it is involved 3-generator groups. In addition, the value of n considered by [5] from the group of order p^n is $n > 2$ where in this research, we focused on the group of order p^4.

A growing body of literature has arises on the potential of the conjugacy class of a group and its application into graph theory. In 2009, Herzog *et al.* [13] linked the commuting graph with the conjugacy classes of groups. The commuting graph is a graph whose vertices are the non-central elements and the edges are connected if their vertices are conjugate to each other. In their paper, the vertices of the graph considered are the non-trivial conjugacy classes of the group. Later, in 2013, Ilangovan *et al.* [14] applied the results of conjugacy classes of the groups of nilpotency class 2 into undirected graph. For the undirected graph, the vertices are the non-central conjugacy classes of a group and their vertices are adjacent if and only if their orders are not relatively prime.

In this research, the classification of 3-generator p-groups, where p are odd primes are considered. The classifications are obtained from Burnside [15] in 1897 who studied the theory of groups of finite order. However, the presentation of the groups have been revised by Ok in [16] and is used in this research.

Throughout this research, by finding the conjugacy classes of 3-generator groups of order p^4, the number of conjugacy classes is derived in general. The conjugacy classes are computed based on the definition of the conjugacy class itself. The results are then connected to some graphs related to conjugacy class.

1.3 Problem Statement

Research on conjugacy classes has become more extensive from year to year. For many years, some of the researchers studied on lower bound and upper bound of conjugacy classes. Interestingly, some of researchers are started to find the exact number of conjugacy classes instead of the upper bound and lower bound. For example, the formula of the exact number of conjugacy classes have been found for dihedral group, alternating group and symmetric group. Not only that, the exact number of conjugacy classes of 2-generator p-group is also studied.

From the previous research, it can be seen that the exact number of conjugacy classes are only restricted to certain groups and order. Hence, this research is an extended version for the case 2-generator p-groups by increasing the number of generators and the orders. In this research, the exact number of conjugacy classes of 3-generator groups of order p^4 is studied. However, the group presentation used for this group is difference from the 2-generator group since the number of generators is different. Still, the exact number of conjugacy classes is limited to 3-generator of order p^4. But, at least the exact number of conjugacy classes for the higher order have been generalized.

In this research, the exact number of conjugacy class is important in order to see the connections between the group theory and graph theory. From the exact number of conjugacy classes, the vertices and the edges of some of the graphs especially in this research can be found. That is the reason why the exact number of conjugacy classes need to compute first in the first part of the results. Then, the results in the computation will be applied into some type of graphs discussed in this work.

1.4 Objectives of the Research

Let G be a 3-generator group of order p^4 and the set Ω is defined as $\Omega = \{(a,b) \in G \times G : lcm(|a|,|b|) = p, ab = ba, a \neq b\} \backslash \{(b,a)\}$. The objectives of this research are:

1. To compute the number of conjugacy classes and find the conjugacy class graph for H_1, H_2 and H_3.
2. To generalize the exact number of conjugacy classes of H_2 and H_3.
3. To determine the generalized conjugacy class graph and the orbit graph for H_1, H_2 and H_3.
4. To introduce two new graphs related to conjugacy classes, namely the generalized commuting conjugacy class graph and the generalized non-commuting conjugacy class graph, for the case $p = 3$.

1.5 Scope of the Research

This research consists of two parts. The first part is in determining the conjugacy classes of 3-generator groups of order p^4 where p are odd primes. The computation of the conjugacy classes is divided into two sections. Firstly, the conjugacy classes are found by using the definition of conjugacy class which considered single element i.e $cl(x) = \{gxg^{-1} : g \in G\}$. In the second section, the conjugacy classes are computed by defining the set omega. The set omega is defined as $\Omega = \{(a, b) \in G \times G : lcm(|a|, |b|) = p, ab = ba, a \neq b\}\backslash\{(b, a)\}$ motivated on the work done by [17] but which only cover the elements of (a, b) excluding (b, a), (a, a) and (b, b). From the definition, the elements considered in the computation of the conjugacy classes for this section are in the form of ordered pairs. Thus, the conjugacy classes are found by using the formula, $cl(\omega) = cl(a, b) = \{g(a, b)g^{-1} : g \in G\}$.

The second part of the research is related to graph theory. The computation of conjugacy classes in the first section is applied in forming the conjugacy class graph. Meanwhile, the general form of the number of elements of ordered pairs and the number of conjugacy classes which are derived in the second section are applied into the orbit graph and generalized conjugacy class graph, respectively. Besides that, two new type of graphs namely the generalized commuting conjugacy class graph and the generalized non-commuting conjugacy class graph are introduced.

1.6 Significance of the Research

Recently, there are many researchers who are interested in studying the relationship between group theory and graph theory. The conjugacy class is one of the properties of group theory that can be studied. The vertices of a graph can be formed from the elements in the conjugacy classes and the edges connect the vertices according to specific rules. The vertices and edges are the most important thing in order to form a graph.

The major contribution in this research is in the group theory itself. Since the exact number of the conjugacy classes of 3-generator groups of order p^4 is found in general in Chapter 4, hence, it can be used directly without manual calculation.

Furthermore, the research also contributes in the findings of the four types of graphs discussed in this research.

1.7 Research Methodology

This research explored the exact number of conjugacy classes of some finite groups. It began by choosing 3-generator groups of order p^4 as the scope. The presentation was given by Burnside [8] and had been revised by Ok [9]. Since this research is focused on 3-generator groups of order p^4, three types of classification are chosen.

This thesis is divided into two parts which are the conjugacy classes of 3-generator groups of order p^4 and their applications on some type of graphs. The first part, on finding the conjugacy classes, is also divided into two phases. In the first phase, the elements of the group are first determined. The conjugacy classes of the group are calculated by using the formula, $\{g(x)g^{-1} \in G$ for all g in $G\}$. The definition of conjugacy class graph is used in which the number of conjugacy class and the center of the group are needed to form the graph.

In the second phase, the set Ω is defined as $\Omega = \{(a,b) \in G \times G : lcm(|a|,|b|) = p, ab = ba, a \neq b\}\backslash\{(b,a)\}$. The conjugation action of the group on the set Ω is involved in the computation. The computation began with determining the elements with order p in each group. Next, the elements that satisfied the condition in the set Ω are identified in order to find the number of elements of ordered pairs in the set Ω. Afterwards, the number of conjugacy class in the set Ω was obtained by applying the definition of the conjugacy classes in the computation i.e $\{cl(\omega) = cl(a,b) = g(a,b)g^{-1} : g \in \Omega\}$. Acknowledge that, some basic concepts of arithmetic sequence and arithmetic series are also used to simplify the number of elements of ordered pairs and the number of conjugacy class in general form.

The second part of this thesis is also divided into two subparts. In the first subpart, the results of the computation of the conjugacy class are applied into graphs namely the orbit graph and the generalized conjugacy class graph. The vertices are determined from the elements in the set Ω while the edges are obtained based on the rules stated in the definition of the graphs.

Two new graphs are defined in the second subpart of this thesis namely the generalized commuting conjugacy class graph and the generalized non-commuting conjugacy graph considering only a special case of $p = 3$. Using similar manner, the results of the computation, specifically on the number of ordered pairs and the number of conjugacy class are applied into these new graphs. The research methodology is shown in Figure 1.1.

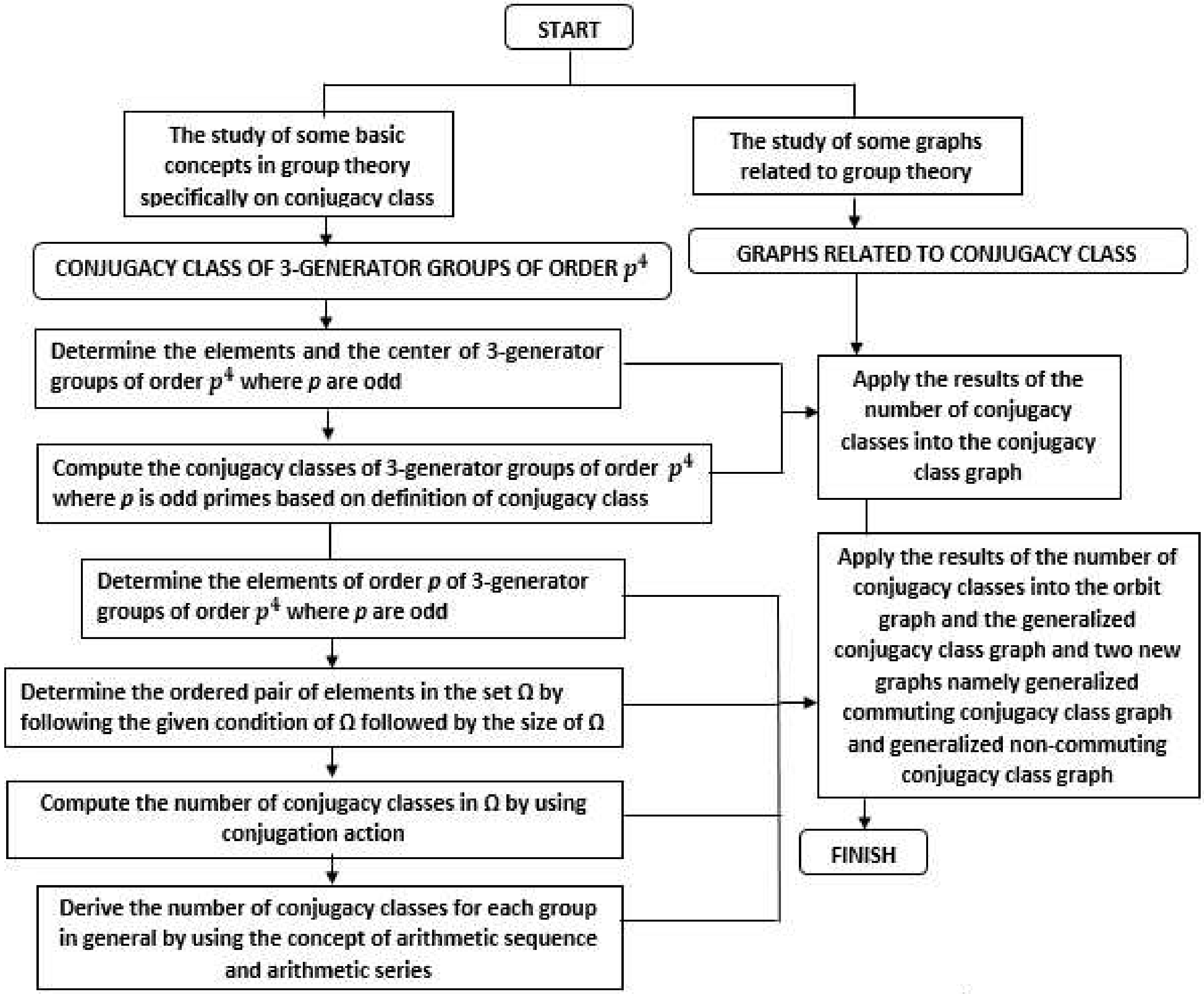

Figure 1.1 Research Methodology

1.8 Thesis Organization

This thesis consists of seven chapters. In the first chapter the introduction to the research has been provided, which include background of the research, problem statements, objectives of the research, scope of the research, significance of the research, research methodology and thesis organization.

In Chapter 2, the literature review of this research is provided. Some basic definitions on group theory and graph theory are included. The classification of 3-generator groups of order p^4 is also presented. In addition, some previous researches on general formula of conjugacy classes of finite group are discussed.

In Chapter 3, the computation of the conjugacy class of 3-generator groups of order p^4 based on definition where p is an odd prime is presented. After the conjugacy classes has been computed, the center of the group is identified in order to fulfill the condition in the definition of the conjugacy class graph. The conjugacy class graphs for each graph are presented in the form of theorems.

In Chapter 4, the results of conjugacy classes on the set Ω are shown. The computation began by finding the elements of the groups with order p. Then, the number of elements of ordered pairs and the number of conjugacy classes found in the set Ω are determined. The steps taken are demonstrated in few lemmas and theorems.

Next, the graphs associated to the conjugacy classes in Chapter 4 are illustrated in Chapter 5. The graphs considered are the orbit graph and the generalized conjugacy class graph. The number of elements of ordered pairs and the number of conjugacy class found in the set Ω are applied into these two graphs, respectively. The vertices of the graphs are identified from the elements in the set Ω and the edges are formed when the rules in the definition of each graph are followed. Several theorems are presented to explain the graphs.

In Chapter 6, two new graphs are introduced based on the computation of the conjugacy classes as well as the determination of some related graphs in the previous chapter. The graphs are called the generalized commuting conjugacy class graph and the generalized non-commuting conjugacy class graph. The value of the odd prime considered for this graph is $p = 3$.

The last chapter concludes the overall contents of this thesis. Other than that, some suggestions are given for future research. The thesis organization is illustrated in Figure 1.2.

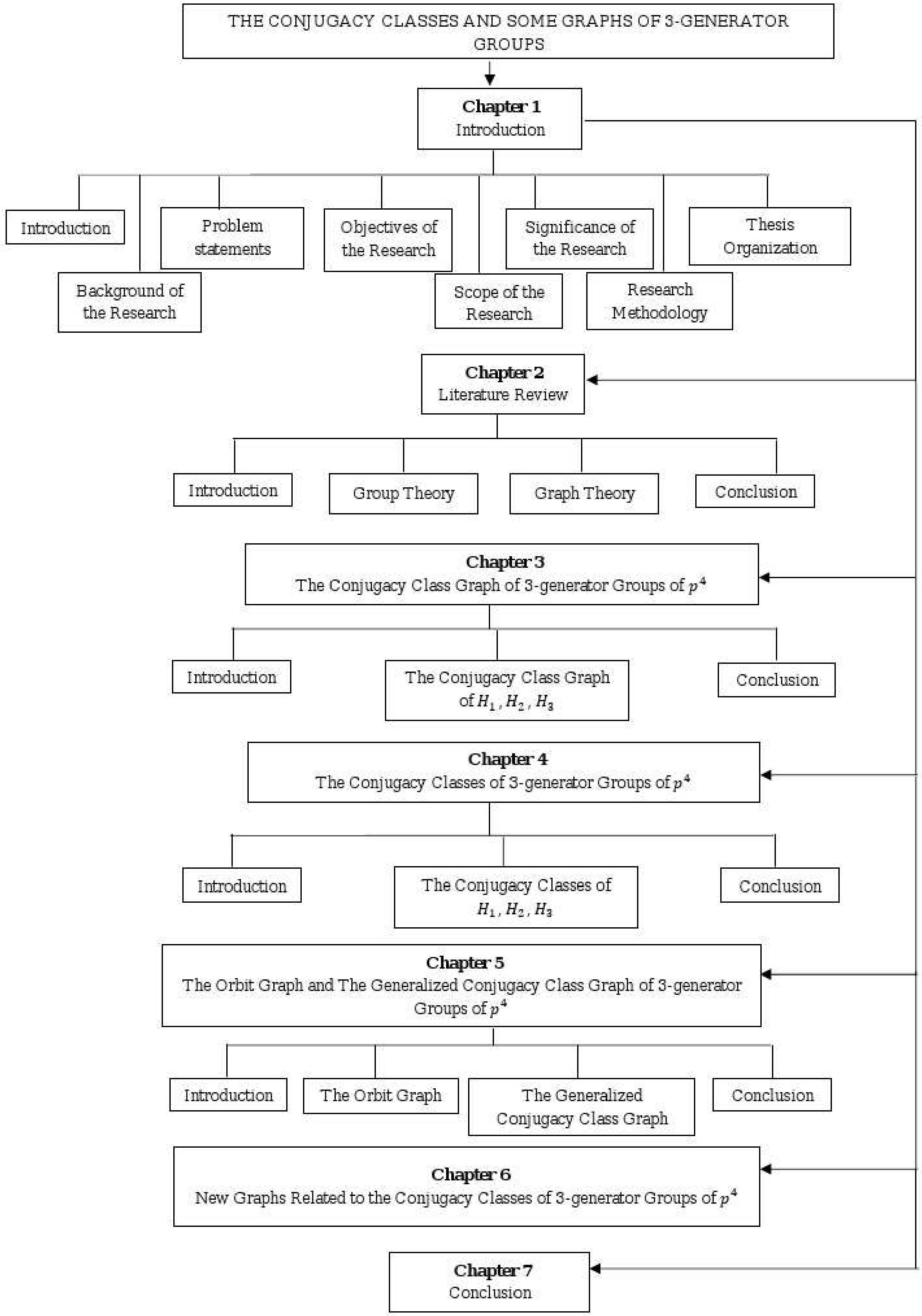

Figure 1.2 Thesis Organization

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LIST OF PUBLICATIONS

Indexed Journal

1. **Alia Husna Mohd Noor**, Nor Haniza Sarmin and Hamisan Rahmat. (2018). Related graphs of the conjugacy classes of a 3-generator 5-groups. Malaysian Journal of Fundamental and Applied Sciences, Special Issue on Natural Science and Mathematics (ESCon2018), 454-456.(Indexed by ISI)
2. **Alia Husna Mohd Noor** and Nor Haniza Sarmin. (2018). On the conjugate and conjugacy class graphs related to a 3-generator 3-group. Advanced Science Letter, 24(6), 4395-4398. (Indexed by WOS)
3. **Alia Husna Mohd Noor**, Nor Haniza Sarmin and Sanaa Mohamed Saleh Omer. (2016). On graphs associated to conjugacy classes of some three-generator groups. Jurnal Teknologi, 79(1), 55-61. (Indexed by ISI)

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1. **Alia Husna Mohd Noor**, Nor Haniza Sarmin and Sanaa Mohamed Saleh Omer. (2016). The conjugacy classes of some three-generator groups. Menemui Matematik, 38(1), 22-26.

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1. **Alia Husna Mohd Noor** and Nor Haniza Sarmin. (2018). On a graph related to the conjugacy classes of 3-generator p-groups. Proceedings of International Conference on Mathematical Sciences and Technology 2018.
2. **Alia Husna Mohd Noor**, Nor Haniza Sarmin, Siti Norziahidayu Amzee Zamri and Hamisan Rahmat. (2018). On the orbit graph of some 3-generator 2-groups. AIP Conference Proceedings 1974, 030011(2018);doi:10.1063/1.5041655