CO-PRIME AND RELATIVE CO-PRIME PROBABILITY FOR NONABELIAN METABELIAN GROUPS OF ORDER AT MOST 24 WITH THEIR RELATED GRAPHS

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DEDICATION

This dissertation is dedicated to my beloved abah and mama who have been with me at all times despite the hardship I went through for this dissertation.

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ABSTRACT

The concept of probability involving groups started with a notion known as the commutativity degree of a group. Later, a new definition is introduced namely the co-prime probability of a group. The probability of a random pair of elements in a group G is said to be co-prime when the greatest common divisor of the order of x and y, where x and y are in G, is equal to one. This co-prime probability is then further extended to the relative co-prime probability of a group and it is newly defined in this dissertation. The probability that two randomly selected elements from H and G is called relative co-prime when the greatest common divisor of the order of h and g, where h is in H and g is in G, is equal to one. This dissertation also discusses on the co-prime graph whereby a graph whose vertices are elements of G and two distinct vertices are adjacent if and only if the greatest common divisor of order x and y is equal to one. The study of the co-prime graph is then extended to the relative co-prime graph where the vertices are elements of a group and two distinct vertices are adjacent if and only if their orders are co-prime and any of them is in the subgroup of the group. Past researchers studied the co-prime probability and their related graphs as well as relative co-prime graph on p-groups and dihedral groups but none did on the nonabelian metabelian groups. Hence, this dissertation aims to be more specific by determining both the co-prime and relative co-prime probability together with their related graphs for nonabelian metabelian groups of order at most 24. The number of edges, the types of the graph and the properties of the graph such as the dominating number and the independent number are discussed. Both Maple 2016 software and some related results by previous researches are used in order to achieve the objectives of this dissertation. It is found that for the co-prime and relative co-prime probability for nonabelian metabelian groups of order at most 24, the results varies for each group with different order. As for the co-prime and relative co-prime graph, the results shows that the dominating number for each group is one while the number of edges, the types of graph and the independent number for each group varies with different order.

ABSTRAK

Konsep kebarangkalian yang melibatkan kumpulan bermula dengan darjah kekalisan tukar tertib bagi suatu kumpulan. Kemudian, satu definisi baru telah diperkenalkan iaitu kebarangkalian ko-perdana bagi suatu kumpulan. Kebarangkalian bahawa unsur yang dipilih secara rawak dalam kumpulan G dikatakan sebagai ko-perdana apabila pembahagi umum yang paling besar untuk peringkat x dan y, di mana x dan y dalam G, adalah sama dengan satu. Kebarangkalian ko-perdana ini kemudian diperluaskan selanjutnya kepada kebarangkalian ko-perdana secara relatif bagi suatu kumpulan dan ditakrifkan di dalam kajian ini. Kebarangkalian bagi dua unsur rawak dari H dan G dinamakan ko-perdana secara relatif, apabila pembahagi umum yang paling besar untuk peringkat h dan g, di mana h dalam H dan g dalam G, adalah sama dengan satu. Disertasi ini juga menerangkan berkenaan dengan graf ko-perdana di mana bucu-bucu adalah unsur bagi G dan dua bucu yang berbeza adalah bersebelahan jika dan hanya jika pembahagi umum yang paling besar untuk peringkat x dan y adalah sama dengan satu. Kajian berkenaan graf ko-perdana ini kemudian diteruskan ke graf ko-perdana secara relatif di mana bucu-bucu adalah unsur bagi suatu kumpulan dan dua bucu yang berbeza adalah bersebelahan jika dan hanya jika peringkat bagi unsur-unsur berkenaan adalah ko-perdana dan setiap daripada mereka ada di dalam subkumpulan bagi kumpulan tersebut. Penyelidikan terdahulu membuat kajian berkenaan dengan kebarangkalian ko-perdana dan graf yang berkaitan dengannya berserta dengan graf ko-perdana secara relatif pada kumpulan p dan kumpulan dihedral tetapi tiada kajian terhadap kumpulan metabelian tidak abelian. Oleh itu, disertasi ini bertujuan untuk menjadi lebih khusus dengan menentukan kedua-dua kebarangkalian ko-perdana dan ko-perdana secara relatif bersama dengan graf yang berkaitan dengannya untuk semua kumpulan metabelian tidak abelian dengan peringkat tidak melebihi 24. Bilangan sisi, jenis-jenis graf dan sifat-sifat yang berkaitan dengan graf seperti nombor berdominasi dan nombor bebas dibincangkan. Kedua-dua perisian Maple 2016 dan keputusan berkaitan dari penyelidikan terdahulu digunakan untuk mencapai matlamat disertasi ini. Didapati bahawa kebarangkalian ko-perdana dan ko-perdana secara relatif untuk semua kumpulan metabelian tidak abelian dengan peringkat tidak melebihi 24, keputusannya berbeza bagi setiap kumpulan dengan peringkat yang berbeza. Bagi graf ko-perdana dan ko-perdana secara relatif, keputusan menunjukkan bahawa nombor berdominasi untuk setiap kumpulan adalah satu manakala bilangan sisi, jenis-jenis graf dan nombor bebas untuk setiap kumpulan adalah berbeza dengan peringkat yang berbeza.

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LIST OF SYMBOLS

$Aut(G)$	-	The automorphism of G
$[a, b]$	-	Commutator of a and b
$Cl(a)$	-	Conjugacy class of the element a
$deg(x)$	-	The degree of vertex x
$E(G)$	-	Edge set of G
G	-	A group
G/H	-	Factor group of G by H
$\|G\|$	-	Order of the group G
H	-	A subgroup of G
$H \cap N$	-	Intersection of H and N
$N \rtimes H$	-	Semi direct product of N and H
$P(G)$	-	The commutativity degree of G
$P(H, G)$	-	The relative co-prime probability of G
$V(G)$	-	Vertex set of G
$(\|x\|, \|y\|) = 1$	-	The greatest common divisor of order of x and y is equal to one
$\|x\|$	-	Order of the element x
$Z(G)$	-	Centre of a group G
$\alpha(\Gamma)$	-	The independent number of Γ
$\gamma(\Gamma)$	-	The dominating nunber of Γ
Γ	-	A graph
$\Gamma_{copr}(G)$	-	The co-prime graph of G
$\Gamma_{copr}(H, G)$	-	The relative co-prime graph of G
G'	-	Commutator subgroup of G
D_n	-	Dihedral group of order $2n$
$K_{1,n}$	-	A star graph
K_n	-	A complete graph of n vertices
$K_{m,n}$	-	A complete bipartite graph

$P_n(G)$	-	The relative n^{th} commutativity degree of G
$P_{mul}(G)$	-	The multiplicative degree of G
$P_{subm}(G)$	-	The sub-multiplicative of G
$P_{copr}(G)$	-	The co-prime probabiity of G

CHAPTER 1

INTRODUCTION

1.1 Introduction

A very simple explanation on metabelian group is a group that is close to being abelian whereby every abelian group is metabelian but not every metabelian group is abelian. A group G is called metabelian if there exists a normal subgroup H in G such that both H and the factor group G/H are abelian. A group G is also considered as metabelian if and only if the commutator subgroup of its commutator subgroup, G'' has a trivial solution which is $G'' = \{1\}$ [1].

A simple definition on probability is how likely something is to happen. For an example, after flipping a coin, what is the probability or possibility of the coin landing on tail? Mathematically, the probability of an event to happen is a number of ways it happen divided by total number of outcomes.

In 2017, Abd Rhani [2] has extended the idea of commutativity degree of G which was first discovered by Erdos and Turan [3], to co-prime probability of G. This probability is defined as the probability of two random elements of a group are relatively prime or co-prime. Abd Rhani [2] has also determined the co-prime probability for all p-groups, where p is a prime number and for some dihedral groups of order $2n$, D_n, where n is odd. In this research, the co-prime probability is extended to the relative co-prime probability for further research.

As for the graph part, there are many types of graphs which have been studied

before such as commuting graph by Segev and Seitz [4], non-commuting graph by Raza and Faizi [5], conjugate graph by Bertram *et al* [6], conjugacy graph by Erfanian and Tolue [7] and many more. This research focuses on two graphs studied by Ma *et al.* [8] and Abd Rhani [2] which are the co-prime graph and the relative co-prime graph respectively.

In this research, the co-prime probability for nonabelian metabelian groups of order at most 24 and the relative co-prime probability for cyclic subgroups of nonabelian metabelian groups of order less than 24 with their related graphs are determined.

1.2 Research Background

Many researches have been done on commutativity degree for metabelian group in the past years. For examples, Che Mohd [9], Mohd Ali and Sarmin [10], Abd Halim [11], Mustafa [12], Jaafar [13], Abu Bakar [14], Erfanian *et al.* [15], Abdul Hamid [16], Hassan [17], Abu Bakar *et al.* [18] and Abd Rhani [2]. Then, the commutativity degree is extended to the co-prime probability and the relative co-prime graph of some finite groups, but, none did a research on the co-prime probability and the relative co-prime probability for cyclic subgroups of nonabelian metabelian groups.

As for the graph part, a few researches related to graph have been done by previous researchers such as Williams [19], Erdos and Sarkozy [20], Sarkozy [21], Sattanathan and Kala [22], Ma *et al.* [8], Dorbidi [23], Rajkumar and Devi [24] and Abd Rhani *et al.* [25], but in this research, previous works by Ma *et al.* [8] and Abd Rhani *et al.* [25] will be used.

Therefore, the focus of this research is on both the co-prime probability and the relative co-prime probability for cyclic subgroups of nonabelian metabelian groups

and their related graphs.

1.3 Problem Statement

Past researches show that the study of the commutativity degree for nonabelian metabelian groups has been extended to the relative commutativity degree, the n^{th} commutativity degree, the multiplicative degree and the sub-multiplicative degree of the group. Latest in 2018, Abd Rhani [2] has defined another new probability that is the co-prime probability but she only determined it for some finite groups. As suggested by [2], the research can also be extended to the relative co-prime probability.

For the graph part, the research started when Williams [19] first introduced the prime graph in 1981. Later, it has been extended to the co-prime graph by Ma *et al.* [8] and the relative co-prime graph by Abd Rhani [2]. Besides, other researchers also did some studies on the co-prime graph but with different scope of groups.

Hence, in this research, the target is on determining the co-prime probability for nonabelian metabelian groups of order at most 24 and the relative co-prime probability for cyclic subgroups of nonabelian metabelian groups of order less than 24, together with their graphs as none did a research on these matter. The research questions are as in the followings:

1. What are the co-prime probability and the co-prime graph for nonabelian metabelian groups of order at most 24?
2. What is the definition of the relative co-prime probability?
3. What are the relative co-prime probability and the relative co-prime graph for cyclic subgroups of nonabelian metabelian groups of order less than 24?

1.4 Research Objectives

The objectives of this research are listed as in the followings:

1. To determine the co-prime probability and the co-prime graph for nonabelian metabelian groups of order at most 24.
2. To define the relative co-prime probability.
3. To determine the relative co-prime probability and the relative co-prime graph for cyclic subgroups of nonabelian metabelian groups of order less than 24.
4. To determine the total number of edges, the types and some properties of the graph such as the dominating number and the independent number for Objective (1) and (3).

1.5 Scope of the Study

This research focuses on finding the co-prime probability and the co-prime graph for nonabelian metabelian groups of order at most 24 and also, the relative co-prime probability and the relative co-prime graph for cyclic subgroups of nonabelian metabelian groups of order less than 24.

1.6 Significance of Findings

This research will give some contributions to the field of group and graph theory as this is a continuation and study from the previous researchers. Abd Rhani [2] has extended the commutativity degree of G to a new definition namely the co-prime probability. Therefore, it can be a new exploration especially in the field of group theory, also, it can be beneficial to the future researchers.

Graph theory also can be very useful in real life whereby the graph can be used as a solution to a problem. For example, in 1735, Konisberg bridge problem is solved by using the concept of planar and Eulerian graph. Another example of the application of graph is in business management whereby the star graph is related to a model named hub and spoke model.

1.7 Research Methodology

The research started with the commutativity degree that have been determined by previous researchers for different groups, it is then extended to the co-prime probability by Abd Rhani [2] in 2018. This research begins with understanding the metabelian groups, the commutativity degree, the co-prime probability and the relative co-prime graph that are related to the objectives of this research.

This research is divided into two main sections. The first section is on the co-prime probability. Firstly, the metabelian group of order at most 24 by Che Mohd [9] are used to get the list of groups. Then, by using Maple 2016 software [26], the Cayley tables are generated for all groups of order at most 24. From the Cayley table, the order of the element of each group can be determined as well. Next, by using the definition by Abd Rhani [2], the co-prime probability is determined. As for the graph part, by using the definition by Ma *et al.* [8], the co-prime graph is then constructed.

For the second section, the relative co-prime probability is discussed. The relative co-prime probability is first defined. Then, the relative co-prime probability for cyclic subgroups of nonabelian metabelian groups of order less than 24 are calculated. Later, the relative co-prime graph are determined by using the definition of co-prime graph. The research methodology is summarized in Figure 1.1.

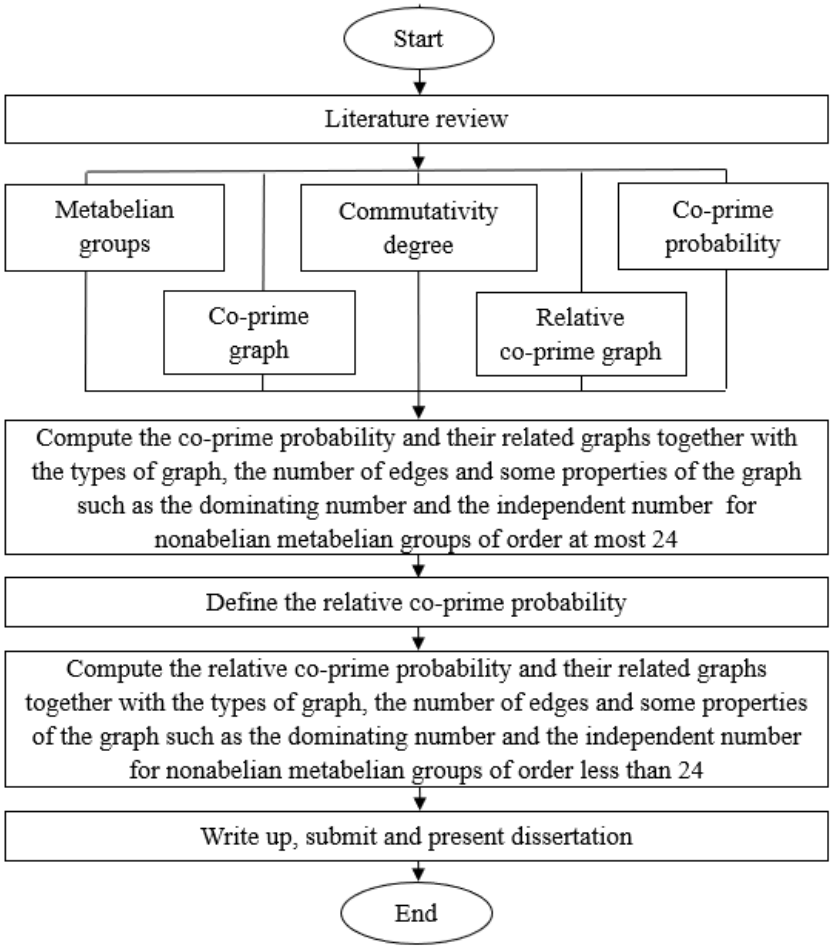

Figure 1.1 Research methodology

1.8 Dissertation Organization

There are five chapters in this research starting with Chapter 1 in which the introduction of the topics of the co-prime probability and its related graphs are explained. The problem statements and the research questions have been pointed out next. The objectives of this research have also been stated besides presenting the scope of the study, significance of the findings and also the methodology of this research.

In Chapter 2, the literature review of the research is discussed. This chapter is divided into two main sections. The first section is on group theory whereby it focuses on the preliminaries on group theory such as theorems, propositions and definitions that are related to the metabelian groups, the co-prime probability and the relative co-prime probability and also some previous researches that have been done on the

commutativity degree of nonabelian metabelian groups. The second section focuses on graph theory. Here, the preliminaries on graphs are stated and the previous works that are related to the co-prime graph and the relative co-prime graph are also discussed.

Next, Chapter 3 focuses on finding the co-prime probability for nonabelian metabelian groups of order at most 24 while in Chapter 4, the relative co-prime probability for cyclic subgroups of nonabelian metabelian groups of order less than 24 are determined. Later, the co-prime and the relative co-prime graph are then discussed to find the total number of edges, the type of graph, the dominating number and the independent number in both Chapter 3 and Chapter 4.

Lastly, in the final chapter which is Chapter 5, the conclusion of the whole research are explained. For future research, some recommendations and suggestions that are related to the co-prime probability, the co-prime graph, the relative co-prime probability and the relative co-prime graph are also included in this chapter.

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