COMPUTING SYLVESTER RESULTANT MATRIX IN HERMITE POLYNOMIALS

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بسم الله الرحمن الرحيم

هذا من فضل ربي

ليبلوني أأشكر أم أكفر

In the Name of Allah, the Compassionate, the Merciful, Praise be to Allah, Lord of the Universe, and Peace and Prayers be upon His Final Prophet and Messenger.

To my Family.

This dissertation is dedicated to them.

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ABSTRACT

There has been increasing interest in the theory of polynomials in different fields of science and engineering. Recent work has shown that enhanced numerical solution can be obtained via expressing polynomials in the orthogonal basis such as the Chebyshev, Legendre or Hermite basis. In some problems, such expression requires transforming resultant matrix between the monomial and the orthogonal or generalized basis. This dissertation concentrates on the possibility of constructing and implementing the Sylvester matrix in the Hermite basis as a computational tool in its orthogonal form. The transformation of the Sylvester resultant matrix between the monomial basis and the orthogonal basis is first studied. The multiplication formulas for some Hermite basis polynomials needed in the computation of the resultant matrix are first derived. Then the computation of the Sylvester resultant matrix in the Hermite basis and the representation of Hermite polynomials with Sylvester type determinants are carried out. The outcomes of this study proved that the Sylvester matrix resultant can be constructed and computed in the Hermite basis. In this form, the matrix can further be applied for working with polynomials in the Hermite basis. Thus. ill-conditioned conversion of polynomials from the orthogonal basis to the monomial basis can be avoided when the input polynomials are represented in the orthogonal basis, in particular the Hermite basis.

ABSTRAK

Minat terhadap teori polinomial dalam pelbagai bidang sains dan kejuruteraan kian bertambah. Hasil kajian terkini telah menunjukkan bahawa jawapan berangka yang lebih jitu dapat diperoleh dengan mewakilkan polinomial menggunakan asas polinomial berortogon seperti asas Chebyshev, Legendre atau Hermite. Untuk masalah tertentu, perwakilan sedemikian memerlukan penjelmaan matriks hasilan dapat dilakukan di antara asas monomial dan asas ortogon atau asas teritlak. Disertasi ini ditumpukan kepada kemungkinan untuk membina dan melaksanakan matriks hasilan Sylvester terhadap asas Hermite sebagai kaedah berpengiraan bagi polinomial yang mempunyai perwakilan asas berortogon. Kaedah penjelmaan matriks hasilan Sylvester di antara asas monomial dan asas orthogon dikaji terlebih dahulu. Rumus pendaraban bagi sesetengah asas polinomial Hermite yang diperlukan dalam pengiraan matriks hasilan tersebut perlu diperoleh terlebih dahulu. Seterusnya pengiraan matriks hasilan Sylvester dalam asas Hermite dilaksanakan dan perwakilan polinomial Hermite dengan penentu Sylvester diperoleh. Hasil kajian ini menunjukkan bahawa pembinaan dan pengiraan matriks hasilan Sylvester bagi polinomial yang mempunyai perwakilan terhadap asas Hermite boleh dilaksanakan. Dalam perwakilan sedemikian matriks tersebut boleh digunakan apabila melibatkan polinomial ke atas asas Hermite. Dengan ini suasana tak sihat pertukaran asas polinomial daripada asas ortogon kepada asas monomial dapat dielakkan apabila polinomial input mempunyai perwakilan terhadap asas ortogon, khususnya asas Hermite.

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CHAPTER 1

INTRODUCTION

1.1 Introduction

Polynomials in the orthogonal or generalized basis arise in many applications such as in linear control theory, polynomial approximation, polynomial interpolation problems, computer-aided geometric design (CAGD) and least square problems. Problems that arise may be represented as polynomials in the orthogonal basis and the solutions are sought in the basis. To be able to solve such problem, basis preserving methods are sought to avoid ill-conditioned basis conversion when solving the problem numerically.

For example, in many computer-aided geometric design (CAGD) problems, it turns out to be crucial, both for numerical stability and for efficiency, to use the Bernstein basis instead of the power series basis. As a result, there is a wealth of CAGD procedures that are formulated entirely in the Bernstein polynomial basis that never require that polynomials be converted to the more familiar monomial basis (Aruliah *et al.*, 2015; Corless, 2004). In solving polynomial equations stable methods do involve representations in the orthogonal or generalized basis (Lim, 2009).

A similar set of basis is available for the Lagrange polynomial basis. These methods rely on the fact that the Lagrange polynomial interpolations can be stably (and efficiently) evaluated in the barycentric form (Higham, 2004). Hermite polynomials are used in the derivation of statistical properties of waves, wave field kinematics and dynamics and wave forces under different conditions. Specifically, covariance functions and approximate spectra are obtained for wave force on vertical cylinder according to Morison's formula (Yuan and Tung, 1984). Hermite basis are used for signal classification and for signal pre-processing. Low frequency acoustic signals are pre-processed using a Hermite orthogonal basis inner product approach. The Hermite pre-processed signals result in feature vectors that are used as input to a parallel bank of radial basis function neural networks for classification (Lowrie, 2006).

1.2 Problem statement

Recently, the problem of mathematical handwriting recognition has been of particular interest. High quality mathematical handwriting recognition can be useful for expression entry and editing in both document processing systems and mathematical software, such as computer algebra systems (Kepner and Gilbert, 2011). Handwritten symbols can be represented as parametric curves approximated in the orthogonal basis. In this representation the possibility of finding all the critical points, loops and cusps can be used to determine features for recognition. These points can be computed using the companion and Sylvester resultant matrix approach. Since the curves are represented in the orthogonal basis, it is preferable to work on the critical points based on the resultant in the orthogonal basis rather than performing ill-conditioned conversions from the orthogonal basis to the power basis.

In relation to this, Alsobhe (2015) had worked on the method of constructing the Sylvester resultant matrix in the Chebyshev polynomial basis. The matrix is then applied to implicitize the parametric curves and also to determine the critical point of the curves. If the resultant matrix of orthogonal polynomials can be determined, the implicit form of the curves and its critical points in the problem of handwriting recognition and other problems can be solved without having to convert the problem to the monomial basis. Besides the Chebyshev polynomial basis, the resultant matrix of other commonly used basis such as the Legendre and Hermite bases can also be determined provided some necessary multiplication operations of polynomials in these bases can be formulated and computed. Since no recent work on resultant matrix of Hermite polynomials can be found, this research aims at determining and computing the Sylvester resultant matrix of polynomials in the Hermite basis. To do this the conversion matrix between the Hermite and monomial basis needs to be constructed and the multiplication formulas related to Hermite polynomials reviewed and formulated.

1.3 Research Objectives

In order to derive and construct the Sylvester matrix for polynomials in the Hermite basis, the following objectives are required:

(i) To determine the conversion matrix from the power series to the Hermite polynomial basis.

(ii) To derive the multiplication formulas for the Hermite polynomials.

(iii) To apply the results of Objective (i) and Objective (ii) to construct and implement the Sylvester matrix in the Hermite basis.

1.4 Scope of the Research

This study focuses on the determination and construction of Sylvester resultant matrix for polynomials in the Hermite basis. The methods of derivation and construction refer to the appropriate approaches that have been obtained for the Chebyshev or the Legendre basis in existing research works.

1.5 Significant of the Study

Resultants are computational tools for determining whether or not a system of polynomials has a common root without actually solving for the roots of these equations. The resultant matrix can be used in solving implicitization problem, finding the GCD, critical points of implicit curves and solving common roots of polynomial systems even though the latter is more suitable for multivariate polynomials with few variables only. In general, the resultant methods can save the computational cost and time in computing the solutions of polynomial theory which are used in different applications of science and engineering. This research gives the avenue of computing the resultant matrix for orthogonal basis polynomials to eliminate the variables set in the polynomial without using power series, unlike the other methods. Hence, the results rely on using an appropriate resultant and companion matrix thereby giving new methods and provide a link with existing work in the theoretical framework of orthogonal basis polynomials.

1.6 Motivation

The researchers usually build models to solve computational problems using polynomial systems. Even if a model is not a polynomial system often it may be reduced to a polynomial system or approximated with a polynomial system. For example, when a system involves the transcendental functions sine or cosine, we may not be able to solve the system directly. Instead we try to replace these trigonometry functions with rational functions or approximate them with polynomials, say with finite Taylor expansions. A well-known technique finite element method is useless without using the approximation of polynomials (Brezzi and Fortin, 2012; Szabó *et al.*, 1991). The reason for these reductions and approximations is that a lot is known about working with polynomial systems. Research on polynomial systems has a wide range of applications in such diverse areas as algebraic geometry, automated geometric design, computer graphics, computer vision, computer algebra, solid modelling, robotics and virtual reality (Agoston and Agoston, 2005; Garcia and Li, 1980). For example, in robotics, when a robot moves, it needs to detect whether it will collide with an obstacle. Both the robot and the obstacles may be represented as polynomials systems. Collision detection is then reduced to solving a polynomial system. To analyse and solve various polynomial systems, mathematicians have developed many effective tools. Resultants are one of the most powerful of these computational techniques.

1.7 Dissertation Organization

This dissertation contains six chapters. These are organized as follows:

Chapter 1 highlights the introduction, summary of the background, research problem, objectives, scope of the study, significance of study, and at the end structural organization of dissertation.

Chapter 2 provides the literature review on resultant and resultant matrices of univariate polynomials, the basic properties of the resultant, the Sylvester and Bezout matrices. A variety of mathematical connections between the Sylvester resultant matrices and the Bezout resultant matrices are explored and a simple block structured matrix that transforms the Sylvester matrix into Bezout matrix is derived. This matrix transformation captures the essence of the mathematical relationships between these two resultant matrices. Basic properties of resultant and power basis transformation of classical polynomials are also explored in this chapter.

Chapter 3 focuses on the methodology that will be applied in this research, details the orthogonal bases representation of classical polynomials and in particular the Chebyshev, the Legendre polynomials and the Hermite polynomials. The theoretical basis of these polynomials, its orthogonality properties, recurrence relations and their generating functions have been discussed. The basis of construction of the Sylvester matrix for the Chebyshev polynomials is reviewed and presented by giving an outline of the method.

Chapter 4 is the main contribution of the research. In this chapter the Hermite basis transformations are investigated with the help of examples. The coefficients conversion algorithm from polynomial in power basis to polynomial in Hermite basis is also presented in this chapter. The multiplication formula for the Hermite polynomials is derived. The implementation of multiplication formula will be applied in the construction of the Sylvester resultant matrix for the Hermite polynomials.

Chapter 5 presents the construction of the Sylvester resultant matrix in Hermite basis. The resultant of two polynomials is discussed in this chapter. The computation of resultant matrix in the Hermite basis that enables the transformation of the polynomials to the power basis is also investigated. The computation of Sylvester matrix in the power basis and their conversion to the Hermite basis is discussed. The transformation of the Sylvester resultant matrix between the power and orthogonal bases is emphasized in this chapter.

Chapter 6 states the conclusion drawn from the current work and suggests possible directions for the future work.

1.8 Conclusion

This chapter presents the background of the research problem, basis preserving polynomial operations, especially the Sylvester resultants in the Hermite basis. Problem formulation identifying research gaps and future calls of previous studies is discussed. The objectives of research based on problem statement are identified to fill theoretical gaps and achieve the research aim in the problem statement. Significance, scope of the study, motivation and organization of dissertation are also enclosed in this chapter.

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