SOME VARIATIONS OF THE COMMUTATIVITY DEGREE OF SOME GROUPS AND THEIR APPLICATIONS IN GRAPH THEORY

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To all my family and friends with love.

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ABSTRACT

Studying the properties of groups based on some probabilistic methods is an appealing branch of research in group theory. It started by investigating commutativity for symmetric groups, and later grew to a massive number of concepts that measure certain aspects of commutativity for finite and infinite groups. Commutativity degree is defined as the probability that two arbitrary chosen elements of the group commute. This concept has been generalized in several ways, one of these generalizations is the probability of an element of a finite group to fix an element of a finite set, namely the action degree of finite groups. In this thesis, the concept of action degree of finite groups is considered where some inequalities and limiting conditions are determined. Moreover, the definition is extended to the infinite case where the action degree of finitely generated groups is presented along with some bounds of this probability. In a different direction, this research presents a new variation of the commutativity degree of groups, namely the order commutativity degree which is the probability that two elements of the same order of the group commute. This commutativity is proven to be equal to one if and only if the group itself is abelian. The order commutativity degree of finite groups is calculated by dividing the cardinality of the set of all pairs of commutative elements of the same order in the group by the cardinality of the set of all pairs of elements of equal order. To simplify the computations, two formulas are provided to compute the order commutativity degree of finite groups. The first one is by using the number of elements of the same order and the sizes of the centralizers and conjugacy classes for some representative elements, while the second formula depends on a newly defined concept called the order centralizer of elements. Additionally, some explicit formulas are provided to calculate the order commutativity degree for certain types of finite groups which are dihedral groups, generalized quaternion groups, groups of composite order, and some groups of prime power order. Later, the order commutativity degree is associated to define a new graph called the order commuting graph in which the vertices of the graph are the elements of a particular order of the group and two vertices are linked by an edge provided that their corresponding elements in the group commute. Moreover, the order commuting graphs are considered and obtained for the previously mentioned groups. These graphs are found to be either empty graphs, complete graphs or bipartite graphs. Finally, several algebraic properties of these order commuting graphs are determined including the degrees of the vertices, graphs independence number, chromatic number, clique number, diameter and girth.

ABSTRAK

Mempelajari sifat-sifat kumpulan berdasarkan beberapa kaedah kebarangkalian merupakan satu cabang yang menarik di dalam kajian teori kumpulan. Ia bermula dengan kajian kekalisan tukar tertib bagi kumpulan simetri dan kemudian berkembang dengan penghasilan banyak konsep yang mengukur aspek tertentu untuk kekalisan tukar tertib bagi kumpulan-kumpulan terhingga dan tak terhingga. Darjah kekalisan tukar tertib ditakrifkan sebagai kebarangkalian bahawa dua unsur yang dipilih secara rawak dari kumpulan adalah berkalis tukar tertib. Konsep ini telah diitlakkan dengan pelbagai cara, salah satu daripada pengitlakan tersebut ialah kebarangkalian sesuatu unsur di dalam suatu kumpulan terhingga menetapkan suatu unsur di dalam set terhingga, yang dinamakan darjah tindakan kumpulan terhingga. Dalam tesis ini, konsep darjah tindakan kumpulan terhingga telah dipertimbangkan, di mana beberapa ketidaksamaan dan syarat batasan telah ditentukan. Tambahan pula, penakrifan tersebut telah diperluas kepada kes tak terhingga di mana darjah tindakan bagi kumpulan yang dijana secara terhingga telah dikemukakan bersama beberapa batasan bagi kebarangkalian tersebut. Dalam arah yang berbeza, kajian ini mengemukakan satu variasi baru kepada darjah kekalisan tukar tertib bagi kumpulan yang dinamakan darjah kekalisan tukar tertib peringkat iaitu kebarangkalian dua unsur berperingkat sama di dalam kumpulan tersebut adalah kalis tukar tertib. Kekalisan tukar tertib ini terbukti sama dengan satu jika dan hanya jika kumpulan itu sendiri adalah abelian. Darjah kekalisan tukar tertib peringkat bagi kumpulan terhingga dihitung dengan membahagikan kekardinalan set yang mengandungi semua pasangan unsur-unsur kalis tukar tertib berperingkat sama di dalam kumpulan, dengan kekardinalan set yang mengandungi pasangan unsur-unsur berperingkat sama. Bagi memudahkan pengiraan, dua rumus telah diperuntukkan bagi menghitung darjah kekalisan tukar tertib tersebut untuk kumpulan-kumpulan terhingga. Pertama adalah dengan menggunakan bilangan unsur-unsur berperingkat sama dan saiz-saiz pemusat serta kelas-kelas kekonjugatan untuk beberapa unsur perwakilan, manakala rumus yang kedua bergantung kepada satu konsep yang baharu ditakrifkan iaitu peringkat pemusat kepada unsur-unsur. Tambahan pula, beberapa rumus tak tersirat telah diperuntukkan bagi menghitung darjah kekalisan tukar tertib peringkat untuk beberapa jenis kumpulan terhingga iaitu kumpulan dwihedron, kumpulan kuaternion teritlak, kumpulan peringkat subahan dan beberapa kumpulan dengan peringkat kuasa perdana. Kemudian, darjah kekalisan tukar tertib peringkat tersebut dikaitkan dengan suatu graf baharu yang digelar graf berperingkat kalis tukar tertib di mana bucu graf tersebut adalah unsur-unsur sesebuah kumpulan dengan peringkat tertentu, dan dua bucu terpaut dengan satu pinggir dengan syarat unsur-unsur yang sepadan di dalam kumpulan tersebut berkalis tukar tertib. Tambahan pula, graf berperingkat kalis tukar tertib tersebut telah dipertimbangkan dan diperolehi bagi kumpulan-kumpulan yang telah disebutkan sebelum ini. Graf-graf yang ditemui ini adalah sama ada graf kosong, graf lengkap atau graf bipartit. Akhir sekali, beberapa sifat algebra bagi graf berperingkat kalis tukar tertib tersebut telah ditentukan termasuk darjah bucu, nombor tidak bersandar, nombor kromatik, nombor klik, diameter dan lilitan.

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$a\|b$	-	a divides b
$P_S(G,X)$	-	The action degree of a finitely generated group G
$P_S(G)$	-	The action degree of a group G on a set S
$\Gamma_{m,n}$	-	A bipartite graph
$Z(G)$	-	Center of the group G
$C_G(a)$	-	Centralizer of a in G
$\chi(\Gamma)$	-	The chromatic number of the graph
$\omega(\Gamma)$	-	The clique number of the graph
$P_X(G)$	-	The commutativity degree of a finitely generated group G
$P(G)$	-	The commutativity degree of a group G
$[g,h]$	-	The commutator of g and h
G'	-	The commutator subgroup of a group G
$K_{n,n}$	-	A complete bipartite graph
K_n	-	A complete graph of n vertices
$cl(x)$	-	The conjugacy class of x in G
$\mathbb{Z}_n$	-	Cyclic group of order n
$d(v)$	-	The degree of a vertex v
$diam(\Gamma)$	-	The diameter of the graph
D_{2n}	-	The dihedral group of order $2n$
$G\times H$	-	Direct product of G and H
$d(u,v)$	-	The distance between two vertices u and v
$E(\Gamma)$	-	Edges of the graph Γ
$\overline{K}$	-	An empty graph
$\emptyset$	-	An empty set
$\square$	-	End of proof
$\phi(d)$	-	The Euler phi function.
$G\cong H$	-	G is isomorphic to H

Q_{4n}	-	The generalized quaternion group of order $4n$
$girth(\Gamma)$	-	The girth of the graph
Γ	-	A graph
G	-	A group
$\langle x \rangle$	-	Group generated by the element x
$B_S(n)$	-	The growth function of the group G with respect to S
$H \triangleleft G$	-	H is a normal subgroup of G
$H \leq G$	-	H is a subgroup of G
e	-	The identity element of a group
$\alpha(\Gamma)$	-	The independence number of the graph
$\|G : H\|$	-	Index of the subgroup H in the group G
$\cap$	-	Intersection
g^{-1}	-	The inverse of g
$Z_S(G)$	-	The kernel of an action
gH	-	Left coset of H with coset representative g
$l_x(g)$	-	The length function of g
$P_n(G)$	-	The n^{th} commutativity degree of G
$K(G)$	-	The number of conjugacy class of a group G
$O(x)$	-	The orbit of x
$C_o(a)$	-	Order Centralizer of a in G
$P_o(G)$	-	The order commutativity degree of a group G
$\Gamma_o(G, \Omega_i)$	-	The order commuting graph of a group
$\|G\|, \|x\|$	-	Order of the group G, order of the element x
$P_A(G)$	-	The probability that an automorphism fixes a group element
G/H	-	Quotient group of G by H
$P(H, G)$	-	The relative commutativity degree of a group G and a subgroup H
$P_n(H, G)$	-	The relative n^{th} commutativity degree of G and H
π	-	The set of positive divisors of an integer

$S_0\,(G)$	-	The set of fixed points of an action
S^{-1}	-	The set of inverses of elements of S
Ω_i	-	The set of non-central elements of G of order i
$A_S(G)$	-	The set of order pairs $(x, s) \in G \times S$ where $xs = s$
$St_G(s)$	-	The stabilizer of s in G
$\sum$	-	Summation
S_n	-	The symmetric group on n symbols
$\cup$	-	Union
$V(\Gamma)$	-	Vertices of a graph

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CHAPTER 1

INTRODUCTION

Researchers have used various methods for the purpose of classifying groups and analyzing or investigating their properties. One of the methods that has been proven beneficial is through measuring the abelianness of non-commutative groups which is known as the commutativity degree of groups. The commutativity degree of a group is defined as the probability that two arbitrarily chosen elements of the group commute and it has been investigated and generalized by many authors in so many directions. Action degree of finite groups is one of these generalizations that was studied and applied for some finite groups acting on some non-empty sets. This research is interested in determining some bounds on the action degree of finite groups and to extend the definition of action degree to the infinite case where the action degree of finitely generated groups is defined along with some limitation boundaries. Meanwhile, in this research, a newly defined concept of commutativity degree is introduced, namely the order commutativity degree which measures commutativity among elements of the same order in a finite group. This new probability is then determined for some categories of finite groups which are dihedral groups, generalized quaternion groups, groups of order pq, and some other groups of prime power order. In addition, two methods to compute the order commutativity degree of finite groups are provided.

Another approach to study algebraic structures of groups is by associating the elements of the group to a graph. In the branch of graph theory, a graph is a network of points that are connected by lines based on some determined conditions. This line of research has received much attention over the past decades and many interesting results were obtained. This research presents a new graph associated to the order commutativity degree of groups which is the order commuting graph that is defined for elements of the same order in the group. Furthermore, the types of the order commuting graphs of the groups under the scope of research are characterized along with some of their properties.

1.1 Research Background

The application of probabilistic methods to group theory has received much attention in the last fifty years and led to interesting results, which could not be obtained using the usual methods. The initial considerable work in this field was presented by Erdös and Turán in a series of papers starting with [1] on some statistics of the symmetric group, S_n. In one of their papers, particularly in [2], they investigated the number of elements in a symmetric group that commute with each other. This concept was then identified as the commutativity degree of non-abelian groups. The commutativity degree of a finite group is defined as the probability that two arbitrarily selected elements in the group commute. There has been a rapid rise in the use of commutativity degree on classifying finite groups. In a major advance, Gustafson in [3] has derived a formula to calculate the commutativity degree of finite groups using the number of conjugacy classes of the group G divided by the order of G. He also proved that the maximum commutativity degree of a finite non-abelian group is $\frac{5}{8}$, and this degree of commutativity only occurs when the order of the center of the group is equal to one fourth the order of the group.

Later, Sherman [4] generalized commutativity degree using the concept of group actions and defined the probability that an element chosen at random from a group G fixes an element chosen at random from a non-empty set S. This probability is referred to in this research as the action degree of groups and denoted as $P_S(G)$. In his paper [4], Sherman considered $P_A(G)$, the probability that an automorphism fixes a group element, which gives commutativity degree in a special case. He considered only the case for which G is a finite abelian group and A is its group of automorphisms.

Omer *et al.* [5] generalized commutativity degree by defining the probability that an element of a group G fixes a set S in which the group G acts on the set S, where the elements of order two are considered. Later, the probability was found for symmetric groups and alternating groups in [6] and some finite non-abelian groups including metacyclic 2-groups in [7]. El-Sanfaz *et al.* [8] extended the work in [5] by restricting the order of the set S for dihedral groups. In this research, the general definition of action degree, that is, the probability of a random element from a finite group G to fix a random element from a non-empty set S is considered where some inequalities and bounds on the action degree of finite groups are determined. On the other hand, recently, Antolin *et al.* [9] defined the commutativity degree of finitely generated groups and proved that there is a generalization of the result of Gustafson in [3] on the upper bound of commutativity degree of finite groups. Inspired by their work, the action degree of finitely generated groups is defined in the present research which generalizes the definition of Antolin *et al.* [9] for the commutativity degree of finitely generated groups. In addition, some bounds and limiting conditions on this degree are determined.

A growing body of literature has investigated many generalizations on commutativity degree and its applications. For instance, Lescot [10] in 1995 defined multiple commutativity degree. Moghaddam *et al.* [11] defined n^{th} nilpotency degree. Erfanian *et al.* in [12] defined relative commutativity degree and relative n^{th} nilpotency degree. Pournaki and Sobhani [13] studied the probability that the commutator of two group elements is equal to a given element which is called the g-commutativity degree. More recently, Moradipour *et al.* [14] obtained the exact value of the commutativity degree of the generalized quaternion groups, dihedral groups, semidihedral groups and quasi dihedral groups.

The previous concepts are strictly associated with the notion of commutativity degree. However, there are no studies on the commutativity degree of elements of equal order. Accordingly, a new notion called the order commutativity degree of groups which is the probability that two elements of the same order commute among all other elements of the same order in the group is introduced in this thesis.

A graph can be linked to a group or a subset of a group in many ways. One of them is the investigation of algebraic properties of the group using the properties of the graph. The commuting graph of a group in which every two vertices are adjacent whenever they commute was studied by various authors. Iranmanesh and Jafarzadeh in [15] characterized most of the finite simple groups by their commuting graphs. Later, Chelvama *et al.* [16] studied certain properties of the commuting graph of dihedral groups. In this research, a new graph associated to the order commutativity degree, namely the order commuting graph is presented. Moreover, the types of the order commuting graphs and some of the graph properties for dihedral groups, generalized quaternion groups, groups of order pq, where p and q are primes and groups of order p^3, where p is an odd prime, are obtained.

1.2 Problem Statement

Commutativity degree and its generalizations have been a major area of attraction for researchers within the field of classifying finite groups and studying their structures. One of the important generalizations of commutativity degree is the probability of an element of a finite group to fix an element of a finite set, namely the action degree of finite groups. However, it can be seen that there is no work related to determining bounds on the action degree of finite groups. Accordingly, part of the aim of the current research is to further study on the action degree of finite groups where some inequalities and bounds on the action degree of finite groups are determined. Meanwhile, in previous research, the action degree was only defined for finite groups. Therefore, the action degree is defined for finitely generated groups and some bounds on them are determined in this research.

Moreover, it is found that in previous studies on commutativity degree, the commutativity of elements of equal order have not been considered. In light of that, this thesis presents a new variant to investigate commutativity among elements of a group having the same order, namely the order commutativity degree. It is proven that this commutativity equals one if and only if the group itself is abelian.

Associating a graph with a group is fast becoming a key instrument in investigating algebraic structures of groups using different methods to link the vertices of the graph based on certain conditions. Several researches have been done on the purpose of constructing a graph by a group. However, there are no previous studies on graphs related to commutative elements in a finite group having the same order. In this research, a new graph called the order commuting graph is defined such that the vertices are elements of equal order and every two vertices are linked by an edge provided that they commute with each other. Furthermore, the types of the order commuting graphs of all groups under scope are characterized and some of their algebraic properties are obtained.

1.3 Research Objectives

The objectives of this research are stated in the following:

(i) To determine some bounds on the action degree of finite groups in general.

(ii) To investigate the action degree of finitely generated groups and to determine some bounds on this degree.

(iii) To introduce a new notion of the commutativity degree that is the order commutativity degree of finite groups and to compute the order commutativity degree for some classes of finite groups which are dihedral groups, D_{2n}, generalized quaternion groups, Q_{4n}, groups of order pq, and groups of order p^3.

(iv) To associate a new graph to the order commutativity degree of groups called the order commuting graph and to obtain the order commuting graphs of all groups under the scope of this research along with some algebraic properties of these graphs.

1.4 Scope of the Study

This research consists of two parts namely group theory and graph theory. The first part of the thesis focuses on the commutativity degree of groups and one of its generalizations, namely the action degree. The action degree of groups is investigated where some bounds on the action degree of finite groups are determined. Moreover, the action degree of finitely generated groups is defined and some bounds on this degree are given. Meanwhile, the order commutativity degree of finite groups is introduced and computed for several classes of finite groups. The groups under scope on the research are dihedral groups, D_{2n}, generalized quaternion groups, Q_{4n}, groups of order $pq,$ where p and q are primes and groups of order p^3, where p is an odd prime, are obtained.

The second part of this research focuses on graph theory. Specifically, the order commutativity degree of finite groups is then applied to graph theory for all the previously mentioned groups and some graph properties are found.

1.5 Significance of Findings

Although finite abelian groups have been completely classified, but it is known that abelian groups accounts for only a small portion of all groups and that non-abelian groups represents the vast majority of all groups. The commutativity degree and its generalizations are considered important invariants that are used for classifying finite non-abelian groups and many results are possible to be obtained with the assistance of this concept and its generalizations.

One major contribution of this research is to determine some inequalities and bounds on the action degree of finite groups. Furthermore, in this research, the action degree of finitely generated groups is defined and some bounds on this degree are determined which may inspire future work involving the action degree of finitely generated groups.

Another major contribution of this research is to introduce a new notion of commutativity degree that is called the order commutativity degree of finite groups and to compute it for several classes of finite groups. In addition, it is proven that a finite group G is abelian if and only if any two elements of the same order in the group commute.

Moreover, the study of graphs is a significant area of mathematical research. Graphs can be used to model many types of relations and processes in physical, biological, social and information systems. Many practical problems can be represented by graphs, emphasizing their application to real-world systems. In this research, a new graph called the order commuting graph of finite groups is introduced and obtained for the groups under the scope of this study. Moreover, some properties of these graphs are determined which give a new knowledge regarding the structure of these groups and relations between their elements. The results obtained can be applied to other categories of groups and to determine more properties of the order commuting graphs of finite groups.

1.6 Research Methodology

This thesis is divided into two main parts. The first part focuses on some variances of the commutativity degree of groups while in the second part one of these variances is applied to graph theory. Firstly, fundamental and essential concepts of some extensions of the commutativity degree of groups and their associated graphs are studied. Moreover, the first part is also divided into two subparts. In the beginning of the first subpart, a previously defined variant namely the action degree of groups is discussed. By using the definition of the action degree of finite groups and its connection to other concepts such as the stabilizer, the kernel subgroup, and the set of all fixed points, some bounds and relations of this degree are determined. In addition, inspired by previous research on the commutativity degree of infinite groups, the action degree of finitely generated groups is defined in this research with respect to a finite generating set. Furthermore, some bounds related to this degree are presented using some existing concepts and properties.

In the second subpart, a newly defined variant of commutativity degree, namely the order commutativity degree of groups is presented. At first, using some properties of groups, the commutators of their elements and the factorizations of the elements, it is proven that any group G is abelian if and only if any two elements in G of the same order commute. The computation of the order commutativity degree of finite groups based on the definition requires determining the size of the set of all commutable pairs of elements of equal order in the group and the size of the set of all pairs of elements of equal order, which cannot be done for some large order groups. Therefore, by using the concepts of the centralizers of elements and their conjugacy classes, an explicit formula to calculate the order commutativity degree of finite groups is provided. In addition, by a newly defined concept called the order centralizer of elements, another formula is also provided to compute the order commutativity degree of finite groups. Afterwards, the order commutativity degree for groups under consideration in this thesis which are dihedral groups, D_{2n}, generalized quaternion groups, Q_{4n}, non-abelian groups of order pq, where p and q are primes, and non-abelian groups of order p^3, where p is an odd prime, are computed using these formulas and with the assistance of Groups, Algorithms and Programming (GAP) software to obtain the orders of elements, their centralizers and their conjugacy classes. Also, a well known function called the Euler phi function is used to assist on determining the exact number of elements of each order for dihedral groups, D_{2n} and generalized quaternion groups, Q_{4n}.

The second part of the thesis deals with graph theory. In the beginning, a new graph called the order commuting graph is defined where its vertices are non-central elements of the group of the same order and every two vertices are adjacent whenever they commute. Next, the definition of the order commuting graph and the results found based on the order commutativity degree of groups are applied to obtain the types of the order commuting graphs for all groups under the scope of this research. Moreover, some algebraic properties of those graphs which are the degrees of the vertices, the chromatic number, the clique number, the independence number, the girth and the diameter of the graph are determined by using the definitions and some existing theorems on these properties.

Figure 1.1 illustrates the research methodology of this thesis.

1.7 Thesis Organization

This thesis is composed of six chapters which includes the introduction, literature review, the action degree of finite and finitely generated groups, the order commutativity degree of groups, the order commuting graphs associated to the order commutativity degree, and the conclusion.

The first chapter, which is the introduction chapter, gives a brief overview of the thesis which includes the background of the research, problem statement, research objectives, scope of the research, significance of the study, research methodology and thesis organization.

In Chapter 2, the literature review of this research is provided. This chapter is divided into three sections. The first section presents some preliminaries related to group theory. It also gives some definitions and basic concepts of the commutativity degree of G and its generalizations. Several results from the previous researches on many variations of commutativity degree of a group and its generalizations are discussed in this section. In the second section, some terminologies and previous studies on graph theory and its applications are presented. Finally, in the third section, a short note on Groups, Algorithms and Programming (GAP) Software is presented.

Chapter 3 is divided into two sections. The first section is devoted to set some bounds on the action degree of finite groups. Meanwhile, a new definition for the action degree of finitely generated groups is presented in the second section in addition to some limiting conditions on this degree.

Chapter 4 begins by introducing the definition of the order commutativity degree of groups along with some properties of this degree. The first section of this chapter provides an investigation of groups with order commutativity degree equal to one. In the remaining part of the chapter, two methods to calculate the order commutativity degree of groups are presented. Lastly, these methods are used to obtain explicit formulas to compute the order commutativity degree of dihedral groups, generalized quaternion groups, non-abelian groups of order pq, where p and q are primes and non-abelian groups of order p^3, where p is an odd prime.

In Chapter 5, the order commuting graph of groups is defined in the first section. In the subsequent sections, the order commuting graphs are found for all groups under consideration in this thesis which are dihedral groups, generalized quaternion groups, groups of order pq, where p and q are primes and groups of order p^3, where p is an odd prime. Furthermore, some properties for these graphs are obtained including the degrees of the vertices, the clique number, the chromatic number, the independence number, the girth, and the diameter.

Finally, Chapter 6 gives the conclusion of the whole thesis which includes a brief summary of the findings. Moreover, areas of further research are also suggested in this chapter.

The outline of the thesis is illustrated in Figure 1.2.

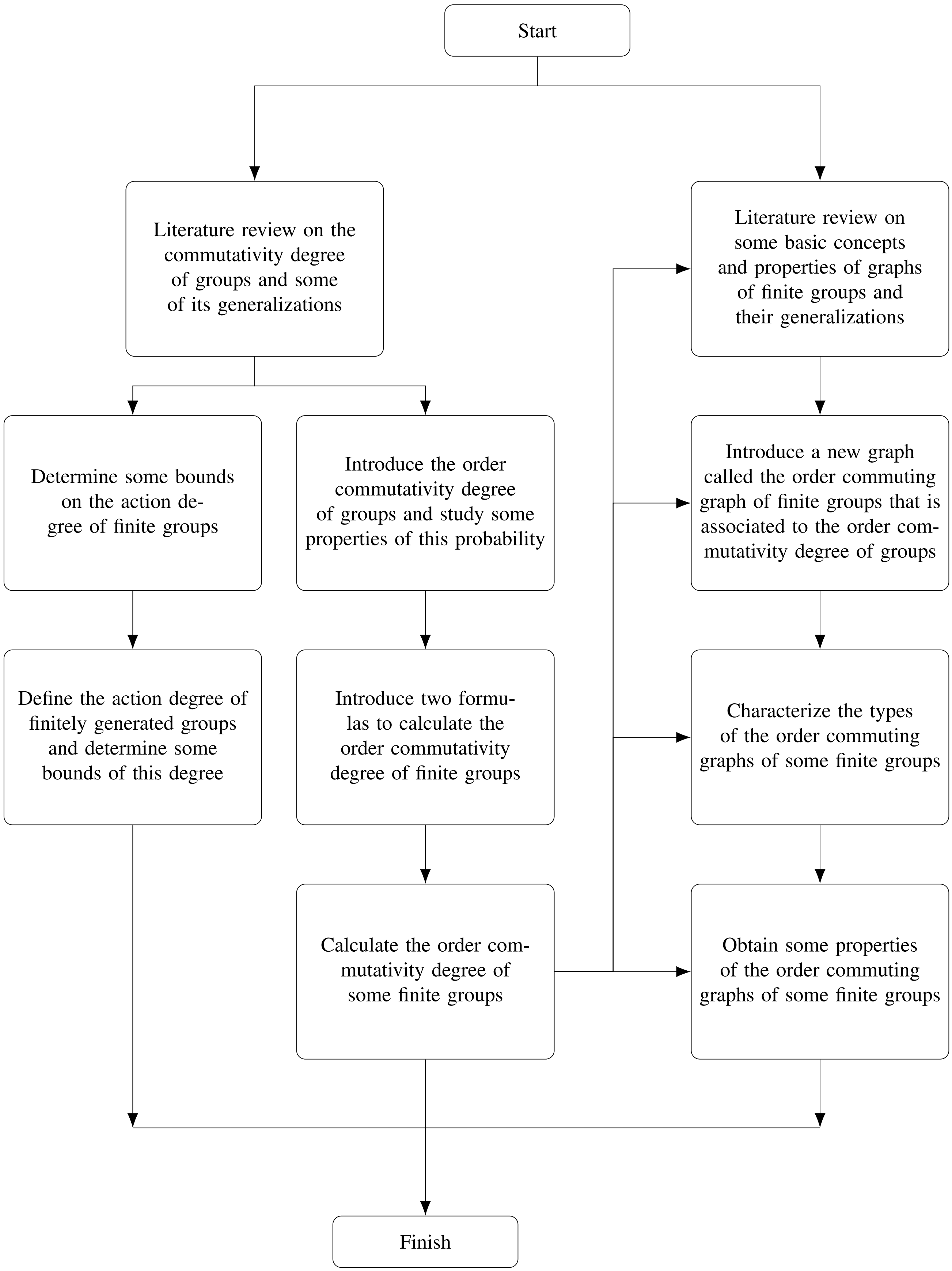

Figure 1.1 Research methodology

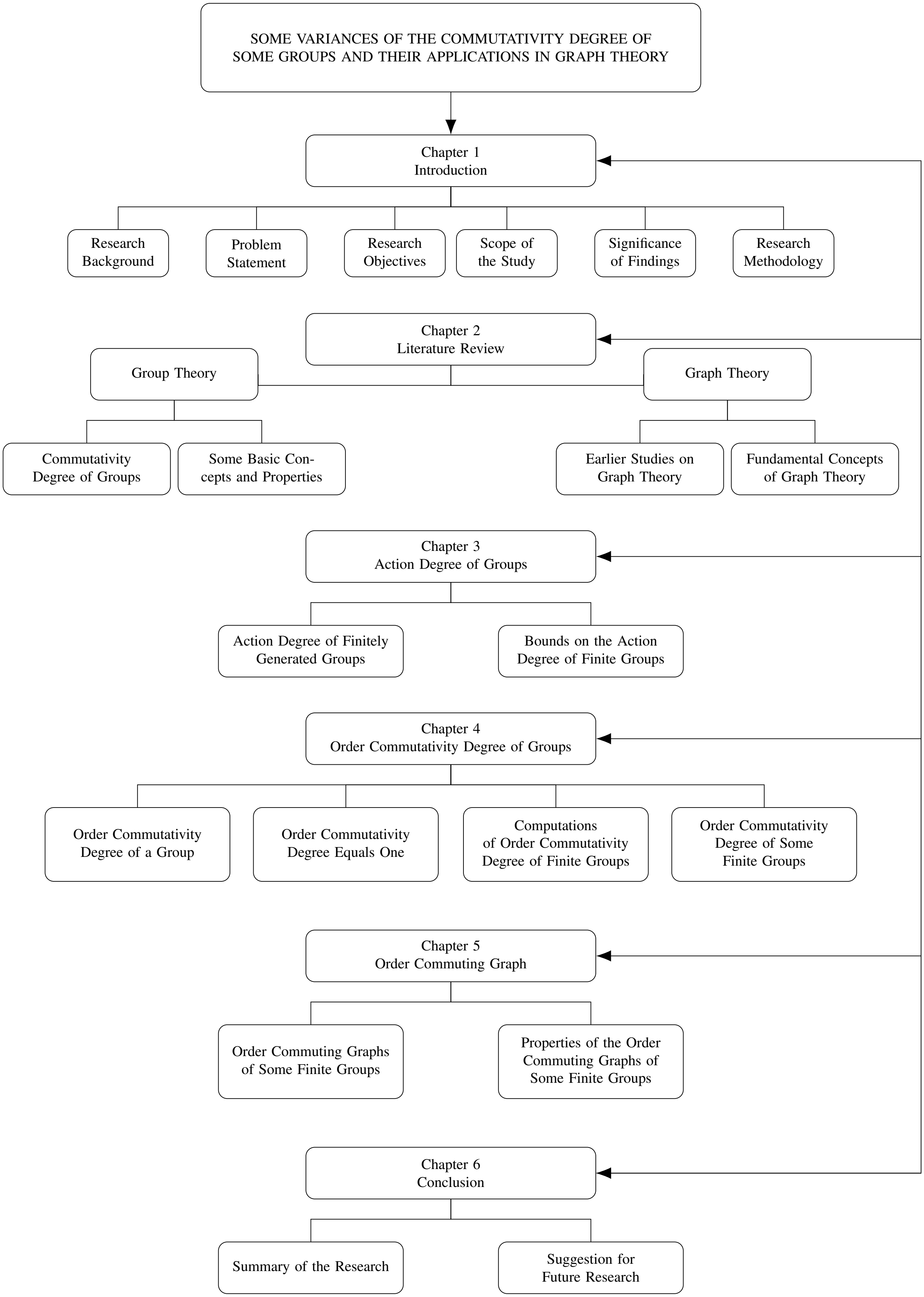

Figure 1.2 Thesis organization

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