SOLVING NONLINEAR SCHRÖDINGER EQUATION OF OPTICAL FIBER TYPE USING INVERSE SCATTERING TRANSFORM

MUHAMMAD FATHULLAH BIN MOHD YUSOF

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Universiti Teknologi Malaysia

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DEDICATION

This dissertation is dedicated to both of my beloved parents for their unconditional love and support.

ABSTRACT

The main purpose of this study is to solve the nonlinear Schrödinger (NLS) equation of optical fiber type using inverse scattering transform (IST) method. Prior to that, simpler problem on the initial-valued Korteweg-de Vries (KdV) equation is discussed to show how such nonlinear evolution equation can be linearized and solved using IST. Then, a more general IST method based on two schemes known as AKNS (Ablowitz-Kaup-Newell-Sigur) and ZS (Zakharov-Shabat) are discussed. AKNS method is described in terms of scattering theory whereas ZS method is expressed solely based on operators. The NLS equation of optical fibre type should be solved using ZS scheme to avoid any specific calculations of the scattering data. Finally, the solutions obtained are used to demonstrate the occurence of solitons from the constructed graph using Mathematica software. It is found that the solution from the NLS equation is a propagating wave enveloped in a wave packet, called a bright soliton. On the other hand, the existence of dark soliton is also detected when the nonlinear term in the NLS equation is negative. Both of these solitons are able to retain its shape after moving over some distance on the graph. The occurence of solitons are able to be demonstrated based on the contructed graphs from the solutions of the NLS equation.

ABSTRAK

Tujuan utama kajian ini adalah untuk menyelesaikan Schrödinger nonlinear (NLS) jenis gentian optik menggunakan kaedah transformasi penyebaran terbalik (IST). Sebelum itu, masalah yang lebih mudah menggunakan persamaan Korteweg-de Vries (KdV) yang mempunyai nilai awal telah dibincangkan untuk menunjukkan bagaimana persamaan evolusi nonlinier tersebut dapat diselaraskan dan diselesaikan menggunakan IST. Kemudian, kaedah IST yang lebih umum berdasarkan dua skema yang dikenali sebagai AKNS (Ablowitz-Kaup-Newell-Sigur) dan ZS (Zakharov-Shabat) dibincangkan. Kaedah AKNS dijelaskan dari segi teori penyebaran sedangkan kaedah ZS dinyatakan hanya berdasarkan operator. Kemudian diputuskan bahawa persamaan NLS jenis gentian optik diselesaikan dengan menggunakan skema ZS untuk mengelakkan pengiraan spesifik dari data hamburan. Akhirnya, penyelesaian yang diperoleh digunakan untuk menunjukkan berlakunya soliton dari graf yang dibina menggunakan perisian Mathematica. Didapati bahawa penyelesaian dari persamaan NLS adalah gelombang penyebaran yang dibungkus dalam paket gelombang, yang disebut soliton terang. Sebaliknya, kewujudan soliton gelap juga dikesan apabila istilah tidak linear dalam persamaan NLS adalah negatif. Kedua-dua soliton ini dapat mengekalkan bentuknya setelah bergerak pada jarak yang tertentu mengunakan graf yang dibina. Kemunculan soliton diperlihatkan berdasarkan graf-graf yang dibina dari penyelesaian persamaan NLS.

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LIST OF ABBREVIATIONS

UTM	-	Universiti Teknologi Malaysia
PDE	-	Partial Differential Equation
KdV	-	Korteweg-de Vries
NLS	-	Nonlinear Schrödinger
IST	-	Inverse Scattering Transform
IMDD	-	Intensity Modulation With Direct Detection
NRZ	-	Non-Return to Zero
SMF	-	Single Mode Fiber
MMF	-	Multi Mode Fiber
ZS	-	Zakharov-Shabat
AKNS	-	Ablowitz, Kaup, Newell and Segur

CHAPTER 1

INTRODUCTION

1.1 Research Background

Partial differential equation (PDE) is a differential equation that constitutes multivariable functions along with their partial derivatives. This type of equation has its origin from the study of geometrical surfaces and classical mechanics. Beginning in the nineteeth century, many mathematicians become so progressively involved in various research problems that can be described by using partial diferential equations. The main reason partial differential equations were used is because they can express many physical laws in nature and in pure mathematics for solving science and engineering problems. Later in its further development, linear PDEs were characterized in order to find the general theory as well as solution methods for these linear equations. This is because PDEs have been found to be significant in the development of both surface theory and to solve the physical problems while these mathematical areas are related by means of variational calculus [1]. With the finding of distribution properties and other fundamental ideas, the theory regarding linear PDE has now been firmly established. However, the subject still has a significant role in the present-day mathematics especially when the nonlinear PDEs are involved. The PDE can be expressed formally in operator form of equation

$$L_x\, u(X) = f(X), \tag{1.1}$$

where L_x is a partial differential operator and $f(X)$ is a given function of independent multivariables $X = (x, y, \ldots)$. If the operator L_x is not linear, then the equation is called a nonlinear PDE. The equation is called a homogeneous nonlinear PDE if $f(X)$ is zero or a nonhomogeneous nonlinear PDE if $f(X)$ is nonzero.

Methods for finding solution of nonlinear equations is just one of several aspects in the theory of nonlinear PDE. However, certain aspects such as uniqueness, existence, and stability of solutions of nonlinear PDEs are of foundational significance. These facts (and probably more) regarding nonlinear PDEs have led this field of study to be applicable to diverse areas of research in mathematical science and engineering.

1.2 Solitary Waves and Their Interactions

Solitary wave (also known later as soliton) was first discovered by J. Scott Russell in Glasgow canal in 1834 where he first called it as 'great wave of translation' and later made a report on his observations to the British Association in 1844 known as 'Report on Waves' [3]. He also did some laboratory experiments to reproduce solitary waves by dropping a weight in water channel. From this experiment, he found that the volume of water displaced by the weight is equivalent to the volume of the generated wave. The speed of the solitary wave, U, can be obtained from

$$U^2 = g(a + h), \tag{1.2}$$

where g is gravitational acceleration, a is a wave amplitude, and h is unaffected water depth. From this equation, we can see that higher waves travel faster [4]. Figure 1.1 shows the parameters and variables applied to describe the solitary wave.

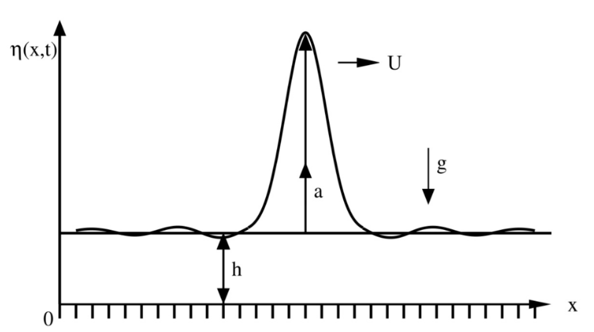

Figure 1.1: Parameters and variables in the soliton [1]

However, there is more being discussed in the Russell's report which is the interaction between waves. What can be analyzed from this result is that if we start with two initial solitary waves moving near to each other, then these waves can later move apart from each other as shown in Figure 1.2 since taller waves have greater velocity. We can see that the taller wave that appear to be initially on the left will overtake the smaller wave and then continue to be moving apart from each other on its own without any change of form [5]. This result has nothing to do with the superposition principle because the process is nonlinear.

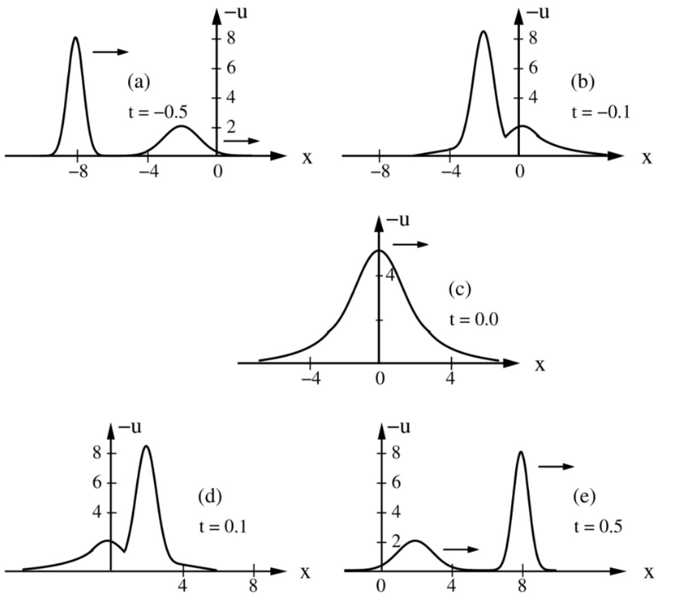

Figure 1.2: Illustration showing the interaction of two solitions at different times [1].

In addition, according to research done by Zabusky and Kruskal [6] in 1965 on the boundary-value problem for the KdV equation, nonlinear waves produced in this case can interact strongly and then continue propagating at later time as if they had been no interaction at all due to some sort of balance between nonlinearity and dispersion. This behaviour of waves led Zabusky and Kruskal to come up with the name 'soliton' (like electron, proton, photon and so forth) to highlight the particle-like character of waves that can retain their shape after the collision.

1.3 Nonlinear Schrodinger Equation of Optical Fiber Type

The communication technology that has been used in daily life nowadays is an industry largely based around optical fiber. With the application of an optical fiber to transmit data in the form of light pulse, this fiber acts to guide the travelling light in it, known as waveguide. Basically what happened in this process is when light goes inside the fiber, it will exit at the fiber's end. The optic fiber is commonly made of glass or silica because of its magnificent clarity [7].

The evolution equation of optical solitons in optic fiber can be expressed by using the governing NLS equation. This equations can be derived from Maxwell's equations [8] to finally yield the form of equation (1.3)

$$i\frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial^2 u}{\partial x^2} \pm |u|^2 u = 0, \tag{1.3}$$

where t is time coordinate, x is spatial coordinate and u is the description for the solitary profile. This equation is our main focus because it can be integrated exactly. As for $\pm$ sign at the nonlinear term, $+$ sign is referring to the self-focusing (bright) while $-$ sign is for self-defocusing (dark) solitons, respectively [9]. This is the form of NLS equation to be solved in this research.

1.4 Problem Statement

It is known that the linear superposition principle is used to solve PDEs if these equations are linear and certain conditions on convergence are met. This principle can generate a single solution by linearly combining a set of solutions. However, the principle cannot be used for nonlinear PDEs. The problem is there has been no generalization made on the method to solve nonlinear PDEs analytically because most techniques developed for linear PDEs are not applicable to it and as a result numerical methods are normally used to obtain their approximate solutions [1]. Therefore, to solve the nonlinear PDEs analytically, transformation of variables is required to linearize these nonlinear PDEs. This can be done by using a method known as the inverse scattering transform (IST). The method can be used to solve NLS other evolution equations and it was invented by [10] known as a Zakharov-Shabat (ZS) scheme that mainly utilized the Lax formulation. Other scheme was also developed by [11] known as the Ablowitz-Kaup-Newell-Segur (AKNS) scheme which is the generalization of the Sturm-Liouville equation.

1.5 Scope of the Study

This study is limited to the KdV equation and ultimately one particular type of nonlinear PDE known as the nonlinear Schrödinger (NLS) equation. To deal with the NLS equation, the consideration is only focus on employing both AKNS and ZS schemes in the IST. The NLS equation studied is the one which is applicable in optical fiber where its solution can produce a particular type of optical solitons known as spatial solitons.

1.6 Research Objectives

The objectives of this research are :

1. To derive the AKNS and ZS schemes in IST method and solve the NLS equation of optical fiber type using the most preferable scheme.
2. To demonstrate the existence of solitons based on the plotted graph from the solution of NLS equation of optical fiber type.

1.7 Significance of the Study

The study seeks to enhance the understanding of optical soliton based on NLS equation of optical fiber type. This is because the equation is derived from the Maxwell's equations that can describe the propagation of light in the optical fiber [8].

Because the NLS equation involves complex nonlinear term, a more general IST method is required to obtain its solution. Thus, this research starts from the discussions in solving the KdV equation will all real terms first, which requires a simpler IST technique. The implementation of the more complex AKNS and ZS schemes in IST can be done more effectively afterwards.

On the other hand, this study also explains the results of both bright and dark solitons based on the plotted graph. This is due to the resulting soliton solution that can propagates differently with time when changing the sign of the nonlinear complex term in the NLS equation. Therefore, these solitons can be used to elucidate different phenomenon (bright or dark) in the way that they propagate.

1.8 Outline of the Research

Chapter 1 begins with the introduction of the research which is background of the research including solitary waves, the form of NLS equation to be used in this research, problem statement, research objectives, scope and significance of the study.

Chapter 2 deals with the literature review on optical fiber. Then the review is done on the solitions in optical fiber. The development of IST is reviewed towards end of the chapter.

Chapter 3 is focused on the derivation of IST. The derivation starts from the earlier method used for solving the KdV equation first since it was the simpler form of evolution equation. Then, the derivation proceeds to a more general IST such as AKNS scheme and ZS scheme. The solution for NLS equation of optical fiber type is finally given at the last section of the chapter.

Chapter 4 discusses the results of both bright and dark soliton obtained from the solution of NLS equations discussed in Chapter 3. The discussion is mainly based on the graphical plot using the Mathematica software.

And lastly, Chapter 5 concludes the discussion and results found in this research. Some suggestions are given for future study research.

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