SOLVING MIXED BOUNDARY VALUE PROBLEMS USING DUAL INTEGRAL EQUATIONS AND DUAL SERIES SOLUTION

ABUBAKAR UMAR

A dissertation submitted in partial fulfilment of the
requirements for the award of the degree of
Master of Science (Mathematics)

Faculty of Science
Universiti Teknologi Malaysia

SEPTEMBER 2020

DEDICATION

This dissertation is dedicated to my parent, for their immense support and financial contributions throughout my studies and who taught me that the best of all knowledge is that which was learned with fashion and dedication. It is also dedicated to my beautiful wife, who taught me that no matter how large the task is, can be accomplished if it is done patiently.

ACKNOWLEDGEMENT

All Praises be to Allah the Lord of the World; prayers and peace be upon to the prophet Muhammad (P.B.U.H). O Allah, to you, belongs all praises for your guidance and care. I would like to also express my sincerest appreciation to my able supervisor Prof. Dr. Ali Hassan bin Mohamed Murid who suggested the dissertation topic and directed the research. I also thanked him for his advice, guidance, and encouragement giving to me during the period of this research.

My deepest gratitude goes to my parents and the family of Ubandawakin R/Shauda for their advice, prayers, and support throughout my studies. My gratitude further goes to my beloved wife Saratu Sa'idu Umar for her endless love, trust, and being with me in any situation. Also, to my fellow graduate students who have assisted me in one way or the other especially Omar Faidullah, Ayub Jama, Sirajo Ibrahim, Lakunti Salisu, and others with their support and contributions.

My fellow postgraduate students at Taman Sri Pulai Perdana should also be recognized for their support and advice. My sincere appreciation also goes to all my colleagues and others who have helped on various occasions. Their views and tips were useful indeed. Unfortunately, it is not possible to list all of them in this limited space. I am grateful to all my friends at Universiti Teknologi Malaysia.

ABSTRACT

Dual integral equations arise when integral transforms are used to solve mixed boundary value problems of mathematical physics and mechanics. A formal technique for solving such equations have been developed. In specific mixed boundary value problems, Fourier transforms are applied, and subsequently, dual integral equations involving Bessel and trigonometric functions have been obtained. The present work aims to consider solvability and solution of systems of dual integral equations involving Fourier transform occurring in mixed boundary value problems for the Laplace's equation with mixed Dirichlet-Neumann boundary conditions. The use of Abel's integral transform was employed. Furthermore, Mathematica software has been used to obtain graphical solutions to the problems.

ABSTRAK

Persamaan kamiran dual timbul apabila jelmaan kamiran digunakan untuk menyelesaikan masalah nilai sempadan campuran fizik matematik dan mekanik. Teknik formal untuk menyelesaikan persamaan tersebut telah dikembangkan. Dalam masalah nilai sempadan campuran tertentu, jelmaan Fourier digunakan, dan seterusnya, persamaan kamiran dual yang melibatkan fungsi Bessel dan trigonometri telah diperolehi. Kajian ini bertujuan untuk mempertimbangkan penyelesaian sistem persamaan kamiran dual yang melibatkan transformasi Fourier terhadap masalah nilai sempadan campuran untuk persamaan Laplace dengan syarat sempadan Dirichlet-Neumann. Penggunaan transformasi integral Abel digunakan. Selanjutnya, perisian Mathematica telah digunakan untuk mendapatkan penyelesaian grafik terhadap masalah tersebut.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

LIST OF ABBREVIATIONS

MBVP	-	Mixed boundary value problem
PDE	-	Partial Differential Equation
DIE	-	Dual Integral Equation
BVP	-	Boundary value problem

LIST OF SYMBOLS

∇	-	Partial Differential (Nabla)
ϕ	-	Phi variant
α	-	Alpha
ξ	-	Xi
η	-	Eta
φ	-	Phi
ω	-	Omega
β	-	Beta

LIST OF APPENDICES

CHAPTER 1

INTRODUCTION

1.1 Problem Background

The mixed boundary value problems (MBVP) are practical situations in most of the potential theory and other mathematical physics problems. The boundary value problem (BVP) for the partial differential equation (PDE) is defined by the mixed boundary conditions on disjoint parts of the boundary of the domain. However, two types of boundary conditions commonly used in solving mixed boundary value problems are Dirichlet and Neumann conditions. The value of the functions is specified by the Dirichlet condition and the Neumann problem is to find a defined, continuous, and differentiable function u over a closed domain D with boundary conditions C that might be represented in an enclosed border. A combination of the function and its derivatives specified at boundary C for the Laplace equation may also be determined by a mixed boundary value problem.

$$\nabla^2 u = 0 \text{ on } D$$
$$u = \left(af + b\frac{df}{dr}\right) \text{ on } C$$

where f is some prescribed function, $\frac{df}{dr}$ is its derivative, a and b are constants, and ∇ is the Laplacian operator. In a wide range of applications, mixed boundary value problems of the potential theory are essential. They can usually be solved best by reducing them to a Riemann Hilbert problem, but there are certain arbitrary constants needed to be interpreted.

The kind of boundary value problems in which the two boundary conditions can be seen at the disjoint part of the boundary is known as the mixed boundary value problem. They can be found in virtually every branch of engineering and are among the hardest to solve. But with the advancement of mathematical software, integral equations have become popular for solving large-scale problems. The first problems of the electrified circular disc and the spherical cap were overcome in the 19th century. The method used can be classified as a Green function method, with the main disadvantage of building up instead of deriving the Green function concerned. The success of the procedure relied primarily on the ingenuity and creativity of the researcher. This way, not many problems can be resolved. In the first half of the twentieth century, many integral transform methods were developed. Some of the first findings have been identified (J. C. Cooke, 1966). Two main methods for the solution of mixed boundary value problems can be found in contemporary literature (Uflyand, 1977).

The problem of axisymmetric needs to be solved using the integral transform method leading to the dual series and dual integral equations. However, if a non-axisymmetric problem required to be solved, each harmonic result must separately be obtained, typically through an extreme lumpy procedure that is becoming more ambiguous because the number of harmonics grows. The new general method systemically introduced by Fabrikant (1989) allowed a non-axisymmetric problem to be solved in a precise and closed manner, for the first time. The new approach also enables non-classical domains to be studied. A wide range of forms have been studied by Fabrikant (1989), and the accuracy of analytical solutions is remarkably high.

The conditions we enforce on the domain boundary are called boundary conditions. The most common boundary condition is to define the function value on the boundary; this kind of condition is known as a Dirichlet boundary condition. For instance, if we specify the Dirichlet boundary conditions for the interval domain $[a, b]$, then we must provide the unknown at the endpoints with a and b; this problem is then called a Dirichlet BVP as shown in Figure 1.1. We have to specify the boundary values along the whole boundary curve in two dimensions and the boundary values on the whole boundary surfaces in three dimensions.

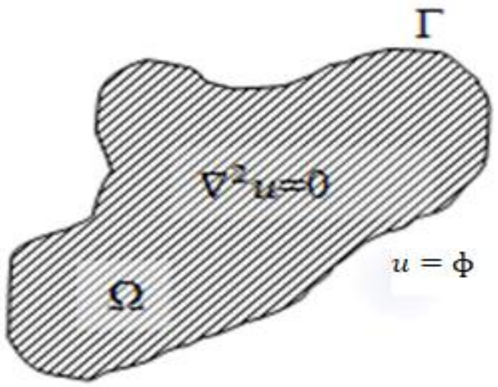

Figure 1.1 Dirichlet problem

The second type of boundary condition is to determine the derivative of the unknown function on the boundary; this type of problem is called the Neumann boundary condition. For instance, if we specify $u'(a) = \alpha$ at the left end of the interval domain $[a, b]$, then we enforce a Neumann boundary condition. If we define only Neumann boundary conditions, then the problem reduces to a pure Neumann BVP as illustrated in Figure 1.2.

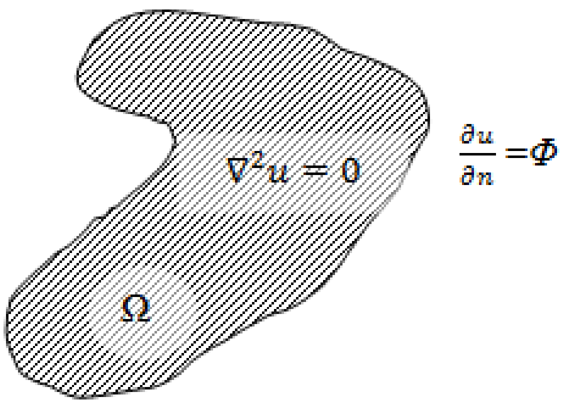

Figure 1.2 Neumann Problem

The Robin boundary condition is the third type of boundary condition known as the boundary condition of the mixed-type, it is a linear combination of the function value and its derivatives at the boundary. For example, we might specify the Robin condition for any unknown $u(x)$ on $[a, b]$ as $u(a) - 2u'(a) = 0$ See Figure 1.3.

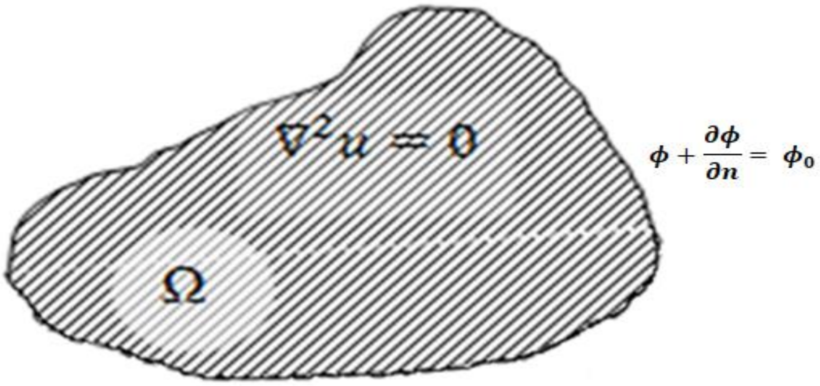

Figure 1.3 Robin Problem

The Zaremba problem is another form of boundary value problem in which the Dirichlet condition is on one side of the boundary and the Neumann condition on the other half of the boundary. The problems of Zaremba are complex and their exact solutions cannot be obtained easily. See Figure 1.4.

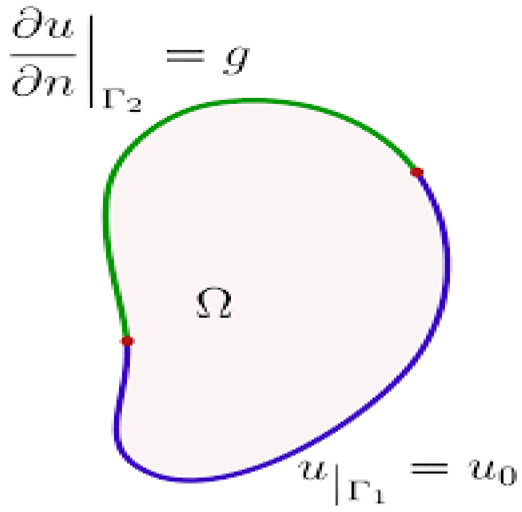

Figure 1.4 Zaremba Problem

Continuum mathematical models also drive the equation of the Laplace. Laplace’s equation occurs in the study of equilibrium processes, including electrostatics, fluid dynamics, thermal conductivity, and diffusion, in a wide variety of physical contexts. The desired (potential) function does not only implement Laplace's equation but also shows some peculiar behaviour in most mathematical problems on the boundary domain (Aghili and Parsania, 2006). Mixed boundary value problems occur and are amongst the most difficult to overcome in many engineering branches (Fabrikant, 1991). However, the potential theory is one of the fields that could represent this theory.

Dual integral equations arise during the development of Integral transform to solve mathematical physics and mechanics problems with mixed boundary value problems (J. C. Cooke, 1966). A formal technique was developed for the solution of such equations. This present work aims to determine the solution of the Laplace's equation using Fourier transforms and dual integral equations method occurring in mixed boundary value problem Duffy (2008), Duffy (2004).

1.2 Problem Statement

In his book, Duffy (2008) presented only brief solutions of how the dual integral equations method can be applied to obtain the solutions of some mixed boundary value problems and used *Matlab* to display the results graphically. In this study, we have shown in detail how to apply the dual integral equations method to solve some mixed boundary value problems and used *Mathematica* to display the results of the analytical solutions for those not familiar with *Matlab.*

1.3 Research Objectives

The objectives of this research are as follows:

i. To solve in detail selected mixed boundary value problems using separation of variables, Fourier transforms, and dual integral equations method.

ii. To write computer codes for the Mathematica software to plot the solutions of the mixed boundary value problems.

1.4 Scope of the Study

There are several methods to solve mixed boundary value problems which include conformal mapping, separation of variables, Wiener-Hopf and Mellin transform methods, and dual integral equations method to solve mixed boundary value problems. This dissertation focuses on the detailed applications of the Fourier transforms and dual Integral equations method to solve some mixed boundary value problems. There are several Mathematical software available such as Matlab, Mathematica, Maple, Sage, and Python. This study focuses on the use of Mathematica software for plotting graphs.

1.5 Outline of the Dissertation

This dissertation is formed into five chapters; the introductory Chapter 1 provides a brief explanation of Laplace's equation and boundary value problems, some discussions on background research, the statement of the problem arising from this research, research objectives, and then review the scope of the study. The methods and applications for solving mixed boundary value problems are outlined in Chapter 2. Chapter 3 covers some auxiliary materials and the approach to dual integral equations. Chapter 4 shows how to apply Fourier and dual integral equations method to solve mixed boundary value problems. Finally, Chapter 5 provides conclusions and suggestions for further studies.

REFERENCES

Abdelrazaq, N. A. (2006). The dual integral equations method for solving nonstationary heat equation under mixed boundary conditions. *Jordan Journal of Applied Sciences - Natural Sciences*, *8*(2), 33–42.

Aghili, A., & Parsania, A. (2006). *in Semi - infinite Strip*. *1*(7), 305–311.

Ahdiaghdam, S., Shahmorad, S., & Ivaz, K. (2017). Approximate solution of dual integral equations using Chebyshev polynomials. *International Journal of Computer Mathematics, 94*(3), 493–502. https://doi.org/10.1080/00207160.2015.1114611

Aizikovich, S. M., Volkov, S. S., & Mitrin, B. I. (2017). The solution of a certain class of dual integral equations with the right-hand side in the form of a Fourier series and its application to the solution of contact problems for inhomogeneous media. *Journal of Applied Mathematics and Mechanics, 81*(6), 486–491. https://doi.org/10.1016/j.jappmathmech.2018.03.018

Babloian, A. A. (1964). Solutions of certain dual integral equations. *Journal of Applied Mathematics and Mechanics*, *28*(6), 1227–1236.

Boyce, W. E., & DiPrima, R. C. (1997). Elementary Differential Equations and Boundary Value Problems, JohnWiley &Sons. *Inc*, *New York, NY*.

Busbridge, I. W. (1938). Dual Integral Equations. *Proceedings of the London Mathematical Society*, *s2-44*(1), 115–129. https://doi.org/10.1112/plms/s2-44.2.115

Chakrabarti, A., & George, A. J. (1994). A formula for the solution of general Abel integral equation. *Applied Mathematics Letters*, *7*(2), 87–90.

Cooke, B. J. C. (1956). *a Solution of Tranter ' S Dual Integral*. *DC*(April 1955), 2–9.

Cooke, J. C. (1966). *I. N. Sneddon, Mixed Boundary Value Problems in Potential Theory (North-Holland Publishing Company—Amsterdam, 1966), viii+282 pp., 80s.* 1966.

Cooke, J. C. (1970). The solution of some integral equations and their connection with dual integral equations and series. *Glasgow Mathematical Journal, 11*(1), 9–20.

Copson, E. T. (1947). On the problem of the electrified disc. *Proceedings of the*

Edinburgh Mathematical Society, *8*(1), 14–19. https://doi.org/10.1017/S0013091500027644

Duffy, D G. (2004). *Transform Methods for Solving Partial Differential Equations*. CRC Press. Retrieved from https://books.google.com.my/books?id=Y6LZV70ZevIC

Duffy, Dean G. (2008). *Mixed boundary value problems*. CRC Press.

Erdélyi, A., & Sneddon, I. N. (1962). Fractional Integration and Dual Integral Equations. *Canadian Journal of Mathematics*, *14*, 685–693. https://doi.org/DOI: 10.4153/CJM-1962-058-6

Erdogan, F., & Bahar, L. Y. (1964). On the Solution of Simultaneous Dual Integral Equations. *Journal of the Society for Industrial and Applied Mathematics*, *12*(3), 666–675. https://doi.org/10.1137/0112057

Eswaran, K. (1990). On the solutions of a class of dual integral equations occurring in diffraction problems. *Proceedings of the Royal Society of London. A. Mathematical and Physical Sciences*, *429*(1877), 399–427.

Fabrikant, V. I. (1989). *Applications of potential theory in mechanics: a selection of new results* (Vol. 51). Dordrecht; Boston: Kluwer Academic Publishers.

Fabrikant, V. I. (1991). Mixed boundary value problem of potential theory in toroidal coordinates. *ZAMP Zeitschrift Für Angewandte Mathematik Und Physik*, *42*(5), 680–707. https://doi.org/10.1007/BF00944766

Fermo, L., & Laurita, C. (2020). A Nyström method for mixed boundary value problems in domains with corners. *Applied Numerical Mathematics*, *149*, 65–82. https://doi.org/10.1016/j.apnum.2019.10.018

Fredricks, R. W. (1958). Solution of a Pair of Integral Equations from Elastostatics. *Proceedings of the National Academy of Sciences*, *44*(4), 309 LP – 312. https://doi.org/10.1073/pnas.44.4.309

Gordon, A. N. (1954). Dual Integral Equations. *Journal of the London Mathematical Society*, *s1-29*(3), 360–363. https://doi.org/10.1112/jlms/s1-29.3.360

Helsing, J. (2009). Integral equation methods for elliptic problems with boundary conditions of mixed type. *Journal of Computational Physics*, *228*(23), 8892–8907. https://doi.org/10.1016/j.jcp.2009.09.004

Hoshan, N. A. (2010). The dual integral equation method for solving the heat conduction equation for an unbounded plate. *Computational Mathematics and Modeling*, *21*(2), 226–238. https://doi.org/10.1007/s10598-010-9067-5

Lowengrub, M., & Walton, J. (1979). Systems of generalized Abel equations. *SIAM Journal on Mathematical Analysis*, *10*(4), 794–807.

Mandal, B. N., & Mandal, N. (1998). *Advances in dual integral equations* (Vol. 400). CRC Press.

Nasim, C. (1986). On dual integral equations with Hankel kernel and an arbitrary weight function. *International Journal of Mathematics and Mathematical Sciences*, *9*.

Ngoc, N. V. A. N., & Ngan, N. T. H. I. (2011). *Solvability of a System of Dual Integral*. *36*(2), 375–396.

Noble, B. (1955). On some dual integral equations. *Quarterly Journal of Mathematics*, Vol. 6, pp. 81–87. https://doi.org/10.1093/qmath/6.1.81

Noble, B. (1958). Certain Dual Integral Equations. *Journal of Mathematics and Physics*, *37*(1–4), 128–136. https://doi.org/10.1002/sapm1958371128

Rappoport, J. M. (2005). *Dual Integral Equations Method for Some Mixed Boundary Value Problems*. 167–176. https://doi.org/10.1142/9789812701732_0012

Sneddon, I. N. (1960). The elementary solution of dual integral equations. *Proceedings of the Glasgow Mathematical Association*, *4*(3), 108–110. https://doi.org/10.1017/S2040618500034006

Srivastav, R. P. (1978). On dual integral equations with trigonometric kernels. *Quarterly of Applied Mathematics*, *35*(4), 524–526. https://doi.org/10.1090/qam/463840

Titchmarsh, E C. (1937). Introduction to the Theory of. *Fourier Integrals*, 200.

Titchmarsh, Edward C. (1948). *Introduction to the theory of Fourier integrals*.

Tranter, C. J. (1950). On some dual integral equations occurring in potential problems with axial symmetry. *Quarterly Journal of Mechanics and Applied Mathematics*, *3*(4), 411–419. https://doi.org/10.1093/qjmam/3.4.411

Tranter, C. J. (1964). An improved method for dual trigonometrical series. *Proceedings of the Glasgow Mathematical Association*, *6*(3), 136–140. https://doi.org/10.1017/S2040618500034900

Uflyand, Y. S. (1977). *Method of Dual Equations in Mathematical Physics Problems*. Nauka, Leningrad.

Walton, J. R. (1975). A Distributional Approach to Dual Integral Equations of Titchmarsh Type. *SIAM Journal on Mathematical Analysis*, *6*(4), 628–643. https://doi.org/10.1137/0506056

Watson, G. N. (1995). *A treatise on the theory of Bessel functions*. Cambridge university press.