HYBRID NUMERICAL APPROACH OF FINITE DIFFERENCE AND ASYMPTOTIC INTERPOLATION METHODS FOR NON-NEWTONIAN FLUIDS FLOW

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A thesis submitted in fulfilment of the
requirements for the award of the degree of
Doctor of Philosophy (Mathematics)

Faculty of Science
Universiti Teknologi Malaysia

MAY 2022

ACKNOWLEDGEMENT

In preparing this thesis, I contacted many researchers and academicians. They have contributed to my understanding and thoughts. I wish to express my sincere appreciation to my main thesis supervisor, Associate Professor Dr Yeak Su Hoe, for encouragement and guidance. I am also very thankful to my external supervisor Dr Norazam bin Arbin and Dr Faisal Salah for their guidance, advice and motivation. Without their continued support and interest, this thesis would not have been the same as presented here.

My fellow postgraduate student should also be recognised for their support. My sincere appreciation also extends to all my colleagues and others who have helped on various occasions. Their views and tips are useful indeed. Unfortunately, it is impossible to list all of them in this limited space.

I am grateful to all my family members, who are my father (Mahadi bin Ismail), my mother (Zainon binti Md Ali), my father-in-law (Saedon bin Busra), my mother-in-law (Juriah binti Md Talib), my brothers and my sisters for their care and encouragement. Thank you very much to my husband (Mohamad Yusuf bin Saedon) and my beloved sons (Mohamad Irfan and Mohamad Arsyad) for their love, strength, and sacrifice.

ABSTRACT

Previous research in the mathematical and physics fields has used computational or empirical approaches to analyse fluid flow problems. Therefore, in this thesis a hybrid numerical approach for non-Newtonian third- and fourth-grade fluid flow problems using the finite difference method and the asymptotic interpolation method are presented. The hybrid method is important for finding accurate results as the size of the problem domain increases to infinity. The finite difference method is used to discretize the nonlinear partial differential equation into a linear system. An asymptotic interpolation method is used to estimate nodal value as the size of the domain tends to infinity. The algorithm is coded using the MATLAB program. A polynomial function that fits the hybrid solution is used to calculate the error of the equation. Theoretical error analysis using truncation error in the finite difference method, right-hand side perturbation linear system, and right perturbation theorem is conducted to determine the norm and range of errors. An implicit numerical scheme of modified fluid problems with an exact solution has been achieved by adding an extra term to the partial differential equation. The norm of error between the hybrid method and exact solution is less than the norm of error between the finite difference method and exact solution. The theory of stability for third-grade fluid is carried out, and the numerical scheme is stable provided that the condition of modulus of the amplifier holds. The hybrid method is used to solve the constant acceleration of an unsteady magnetohydrodynamic third-grade fluid in a rotating frame. The analyses show that the increment of the magnetic and rotating parameters decreases the speed of motion and thus the velocity. The velocity increases with an increase in time. The unsteady magnetohydrodynamic fourth-grade fluid problem in the rotating frame is investigated. Increasing the elastic parameters increases the velocity of the fluid. The problem of heat transfer for third-grade non-Newtonian fluid flow with magnetic effect is addressed. The temperature drops by increasing the Prandtl number. It is noted that increasing the Grashof number increases the temperature and velocity. The obtained results have shown that the hybrid method is consistent, stable, and converges to the solution.

ABSTRAK

Penyelidikan terdahulu dalam bidang matematik dan fizik telah menggunakan pendekatan pengiraan atau empirikal untuk menganalisis masalah aliran bendalir. Oleh itu, dalam tesis ini pendekatan berangka hibrid menggunakan kaedah perbezaan terhingga dan kaedah interpolasi asimtotik untuk masalah aliran bendalir gred ketiga dan keempat tak Newtonan dibentangkan. Kaedah hibrid adalah penting untuk mencari keputusan yang tepat apabila saiz domain permasalahan meningkat ke infiniti. Kaedah beza terhingga digunakan untuk mendiskritkan persamaan pembezaan separa tak linear kepada sistem linear. Kaedah interpolasi asimtotik digunakan untuk menganggar nilai yang tidak diketahui apabila saiz domain permasalahan cenderung ke arah infiniti. Algoritma dikod menggunakan program MATLAB. Fungsi polinomial yang sesuai dengan penyelesaian hibrid digunakan untuk mengira ralat persamaan. Analisis ralat teori menggunakan ralat pemotongan dalam kaedah beza terhingga, sistem linear gangguan sisi kanan dan teorem gangguan kanan dijalankan untuk menentukan norma dan julat ralat. Skim berangka tersirat masalah bendalir diubah suai dengan penyelesaian tepat telah dijalankan dengan menambah sebutan tambahan dalam persamaan pembezaan separa. Norma ralat antara kaedah hibrid dan penyelesaian tepat adalah kurang daripada norma ralat antara kaedah perbezaan terhingga dan penyelesaian tepat. Teori kestabilan bagi bendalir gred ketiga dijalankan dan skema berangka adalah stabil dengan syarat bahawa keadaan modulus penguat kekal. Kaedah hibrid digunakan untuk menyelesaikan pecutan berterusan bendalir gred ketiga hidrodinamik magnet yang tidak stabil dalam bingkai berputar. Analisis menunjukkan bahawa kenaikan parameter-parameter magnetik dan berputar mengurangkan kelajuan gerakan dan dengan itu halaju berkurangan. Halaju bertambah dengan pertambahan masa. Masalah bendalir gred empat hidrodinamik magnet tidak stabil dalam bingkai berputar dikaji. Meningkatkan parameter elastik telah meningkatkan halaju bendalir. Masalah pemindahan haba untuk aliran bendalir tak Newtonan gred ketiga dengan kesan magnet ditangani. Suhu menurun dengan meningkatkan nombor Prandtl. Diperhatikan bahawa peningkatan nombor Grashof meningkatkan suhu dan halaju. Hasil kajian yang diperoleh telah menunjukkan bahawa kaedah hibrid adalah konsisten, stabil dan menumpu kepada penyelesaian.

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LIST OF ABBREVIATIONS

AIM	-	Asymptotic interpolation method
BC	-	Boundary condition
Bvp4c	-	Boundary value solver
CWFD	-	Chebyshev wavelet finite difference
GFDM	-	Generalised finite difference method
GN	-	Gauss-Newton method
FDM	-	Finite difference method
FVM	-	Finite volume method
HAM	-	Homotopy analysis method
HPM	-	Homotopy perturbation method
LBM	-	Lattice Boltzmann method
MHD	-	Magnetohydrodynamic
PDE	-	Partial differential equation
PIA	-	Perturbation-iteration algorithm (PIA)
RK4	-	Runge Kutta fourth-order method
SIF	-	Stress intensity factor
SQN	-	Structured Quasi-Newton method

LIST OF SYMBOLS

u, v, w	-	Component of velocity
$\mathbf{H}$	-	Correction vector
$\mathbf{J}$	-	Current density
$\mathbf{E}$	-	Electric field
n	-	Grade
i	-	Grid point on x −axis
j	-	Grid point on y −axis
$\mathbf{I}$	-	Identity tensor
$\mathbf{b}$	-	Induced magnetic field
J	-	Jacobian
L	-	Length
O	-	Local truncation error
M	-	MHD parameter
a_1, a_2, a_3	-	Parameter in special function
p	-	Pressure
r	-	Radial coordinate
$\mathbb{R}$	-	Real number
$\mathbf{A}_n$	-	Rivlin-Ericson tensors
R	-	Rotating parameter
C_p	-	specific heat at constant pressure
$\mathbf{T}$	-	Stress tensor
T	-	Temperature
$\mathbf{S}_j$	-	Tensor
k	-	Thermal conductivity
t	-	Time
$\mathbf{B}$	-	Total magnetic field
$\mathbf{V}$	-	Velocity field

$\vec{n}$	-	a unit length direction vector
$\|A\|\|A^{-1}\|$	-	Cond(A)
$\nabla \cdot$	-	Divergence
$\Im$	-	Imaginary part
$\|\cdot\|$	-	Matrix norm
$\Re$	-	Real part
$h, \Delta x, \Delta\eta, \Delta\xi$	-	Space grid/ step size at x −axis
$k, \Delta y, \Delta t, \Delta\tau$	-	Space grid/ step size at y −axis

ρb	-	Body force
$\frac{\partial}{\partial t}$	-	Material derivatives
δb	-	Perturbation of b
δx	-	Perturbation of x
ε_i	-	Residual (prediction errors)
μ	-	Viscosity
$\mu\Phi$	-	Viscous dissipation

CHAPTER 1

INTRODUCTION

1.1 Problem Background

Fluid is a substance that includes liquid, gas or plasma that flows under an application of shear stress. Shear stress is a stress state where the force is subjected to a cross-sectional area of a substance. Fluid mechanics is a study of the physics of continual materials which deform when subjected to a force. Fluid dynamics is one of the branches of fluid mechanics concerning fluid movements such as gas or liquid, while fluid statics is the study of fluid at rest. Technological applications of fluid dynamics are like the rocket engine, wind turbine and air conditioning system. Besides that, fluid dynamics provides methods to study ocean currents, weather patterns, plate tectonics and even blood circulation.

1.1.1 Newtonian and non-Newtonian Fluid

Scientists from different fields have studied fluid flow behaviour. There are two types of fluids, namely Newtonian fluids and non-Newtonian fluids. Water and air are examples of Newtonian fluids where the stress is directly proportional to the rate of strain (deformation of material with respect to time). Non-Newtonian fluid, on the other hand, has different characteristics due to constitutive equations. It refers to a fluid in which the viscosity changes depending on the gradient's inflow speed or stress. It is not proportional to the rate of strain, its higher power and derivatives. It also depends on the kinematics history of the fluid element itself (Chhabra, 2010).

Figure 1.1 shows the physical properties of Newtonian and non-Newtonian fluid with power-law index n from shear stress equation $\tau = \mu\left(\frac{\partial u}{\partial y}\right)^n$. The fluid is

Newtonian if $n = 1$. If $n < 1$, the fluid is called pseudo-plastic fluids (or shear-thinning fluids), and if $n > 1$, the fluid is called dilatant (shear-thickening fluids).

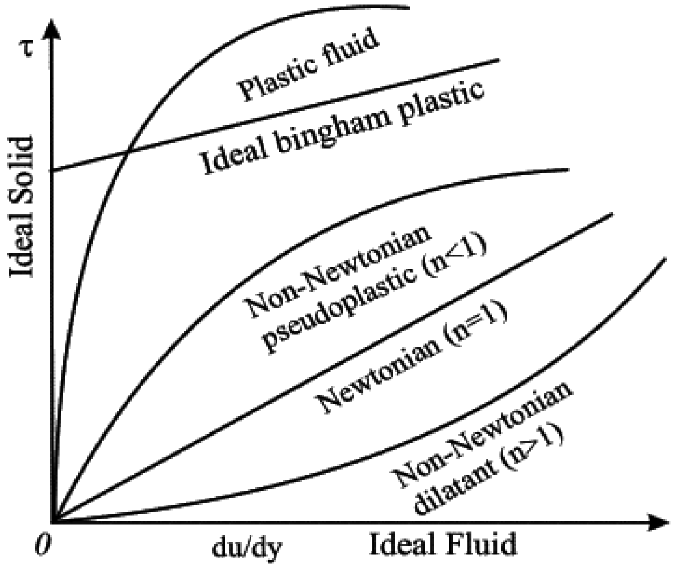

Figure 1.1 Physical properties of the fluid.

The problem in non-Newtonian fluid arises when there is no presence of a single equation that completes the equation to define such fluid. Non-Newtonian fluid has different characteristics in terms of velocity and acceleration. The existence of additional factors such as magnetic fields also offers different effects on the fluid flow. Non-Newtonian fluid flow can be seen in biological fluids such as mucus, saliva and blood.

Furthermore, it can be observed from the engineering and industry such as petroleum, paper production, personal care products like nail polish, and food products such as honey, ketchup, whipped cream, butter, and yoghurt. It also includes natural substances such as lava and magma and other industrial products that have viscoelastic behaviour in their motion. On the other hand, understanding the behaviour of non-Newtonian fluid in landslides is important to prevent disasters (Xiu et al., 2021). Moreover, non-Newtonian fluid is also used in designing body vests for police or the military (Seshagiri et al., 2015).

Normal stress effects that demonstrate fluid elasticity are caused by viscoelastic fluids. Due to the complexity of fluids, it is difficult to characterise non-Newtonian fluids, and there is no single constitutive equation accessible that covers the physical behaviour and properties of all non-Newtonian fluids. As a result, numerous models and constitutive equations have been presented and developed to analyse and examine all non-Newtonian fluid properties.

1.1.2 Differential Types of non-Newtonian Fluid

The mathematical fluid models are classified into differential type, rate type and integral type (Gul et al., 2015; Khan et al., 2015; Khandelwal and Mathur, 2015). Differential type is determined by the derivatives of the local deformation tensor with respect to time. A rate type model is used to describe materials with little memory, such as dilute polymeric solutions. Meanwhile, materials with high memory, such as polymeric melts, are considered integral types (Nazari, 2014).

Differential type is divided into three subclasses which are second-grade, third-grade and fourth-grade. Second-grade fluid is the most basic subclass of non-Newtonian fluid, and it can only describe the normal stress differences. The governing equations for third- and fourth-grade fluids, on the other hand, are substantially more complex, and these fluids can predict shear thickening (viscosity increases with increased stress) or thin (viscosity reduces with increased stress). The constitutive equation for the differential type of non-Newtonian fluid can be seen in Chapter 2, where it shows the relationship between stress and local properties of the fluid.

1.1.3 Magnetohydrodynamics

Magnetohydrodynamics (MHDs), also known as magnetic field fluid dynamics or hydromagnetic, is the study of the dynamics of electrically conducting fluids in the presence of a magnetic field, such as saltwater and electrolytes (Makhiji, 2012). The term MHD is based on magneto, which means magnetic, hydro, which means water or liquid, and dynamics, which refers to the movement of an object caused by forces (Dorch, 2007). MHD is used in various engineering and technological fields, including

MHD power generators, MHD pumps and the petroleum industry. Interactions exist between the motion of the fluid flow, where electric current is generated when fluid flows across magnetic lines and the transverse magnetic lines of forces that contribute to other forces on fluid.

MHD has wide applications; for example, MHD power generating converts thermal and kinetic energy straight to electricity in terms of power generation. This could replace conventional power generation, which typically uses high conversion potential energy to transfer thermal energy to mechanical energy and electrical energy, resulting in increased capital and maintenance costs (Ajith Krishnan and Jinshah, 2013).

MHD laser-powered generators act as a solution to the problem of power generation in space. In addition, it has the potential as a converter to convert space-based lasers to electrical power (Jalufka, 1986). Concerning fluid flow, the problem of high and low rates of velocity while using conventional pumps has led to the development of several types of MHD pumps. This includes seawater pumping, molten metal pumping, molten salt pumping and nanofluid pumping (Al-Habahbeh et al., 2016).

1.1.4 Rotating Frame and Porous Medium

A rotating frame is one that rotates in relation to an inertial reference frame. Because of its significant applications in nature, such as spiral galaxies and atmospheric circulation, the study of fluid flow in a rotating frame has grown tremendously. There are some works in this area concentrating on rotating frames, such as MHD’s fourth-grade rotating flow between two parallel infinite plates (Rana et al., 2012), constant accelerated flow for third-grade fluid in a rotating frame (Aziz et al., 2012) and Stokes’ first problem for third-grade fluid rotating flow (Shahzad et al., 2008).

A porous medium is a substance that has pores, and fluid flow, which has recently emerged as an interesting study area. Because of the wide range of applications in geophysics and engineering, for example, monitoring the subsurface spread of chemical wastes and toxins, the study of MHD flows in porous media with a rotating effect has increased prominently (Imran et al., 2014). Fluid flow in porous media can also be observed in the human body system, like the blood flow via arteries (Eldosky, 2012). Imran et al. (2014), Anita (2015), Garg et al. (2015), Ghani et al. (2016), Parida and Padhy (2018) and Arifuzzaman et al. (2019) previously investigated fluid flow movement through the porous medium.

The study of MHD flow through porous medium with rotational effect has grown in popularity due to various applications in geophysics and engineering, for example controlling the subsurface spread of chemical wastes and pollutants (Imran et al., 2014, Salah et al., 2011, Hayat et al., 2008, Abelman et al., 2009, Hayat and Hutter, 2004, Salah et al., 2013, Hayat and Wang, 2003, and Imran et al., 2014) are some of the studies that deal with the porous medium and rotation.

1.1.5 Heat Transfer

There have been extensive scientific experiments on heat transfer phenomena in non-Newtonian fluid flow due to its significance in many fields, for instance, metallurgical process, production of polymer film, colloidal ceramics processing and plastic manufacture. Furthermore, heat transfer determines the highest and lowest temperature in a system like the in-car radiator or food oven.

Heat transfer modes are classified into three types, namely: conduction, convection, and radiation. The spread of heat caused by temperature gradients is referred to as conduction. Heat conduction (thermal conduction) is a process in which heat is transferred within a body because of particle collision. For example, when the car engine is turned on, the hood warms up due to heat conduction from the engine to the hood. Next, boiling water is an example of heat convection, which refers to the heat transfer through moving fluid and can only occur in fluids and gases. Finally, radiative heat transfer is heat transport by electromagnetic waves such as microwaves. However, in most real situations, these modes will coexist, such as an electric oven.

Previous researchers have investigated many models of heat transfer in fluid flow. For example, Arifuzzaman et al. (2019) examined the heat transfer flow of a fourth-grade radiative. Their research concludes that as the parameters of the second-, third-, and fourth-grades grow, the temperature will rise. Moreover, Uddin et al. (2019) examined heat transfer-induced natural convection in a vertical oscillating cylinder. According to their findings, fluid temperature is reduced because of the thickening of the thermal boundary layer. Furthermore, heat transfer analyses on MHD third-grade fluid could be found in Baoku et al. (2013), Aiyesami et al. (2012) and Sajid et al. (2007).

1.1.6 Numerical Methods

Non-Newtonian fluid equations are mostly complex, demanding and need appropriate methodologies for problem-solving. In this case, an analytical method of homotopy analysis methods (HAM) can be used to address the fluid flow problem (Sajid et al., 2006; Hayat et al., 2011; Aziz et al., 2012; Shafiq et al., 2013). In addition, other studies applied the Fourier sine transform and the Laplace transform to obtain an exact solution (Salah et al., 2013; Tan and Masuoka, 2005; Hayat et al., 2008; Hayat and Hutter, 2004; Salah et al., 2011; Khan et al., 2011, Ali et al., 2012; Eldesoky, 2012). However, the analytical method is ineffective if the problem system is more complex (Loredo et al., 2016).

In many engineering applications, numerical methods can handle massive systems, nonlinear equations and intricate geometries. Therefore, numerical methods have been widely used, including the finite difference method (FDM) (Islam et al., 2011), the finite difference with successive under relaxation (Hayat and Wang, 2003; Rana et al., 2012), FDM on a 3D-staggered grid (Tomé et al., 2002; Tomé et al., 2004; Tomé et al., 2008), Newton method (Shahzad et al., 2008), finite element method (Sajid et al., 2008), generalised finite difference method (GFDM) (Muelas et al., 2019) and implicit finite difference of the Keller-Box method (Rawi et al., 2020).

The FDM is the oldest method that divides space and time coordinates into a rectangular grid and could represent the model's accuracy. This method is suitable for solving partial differential equations (PDEs), including linear or nonlinear, dependent and time-dependent problems, and different boundary conditions (BCs). Many numerical solution techniques to solve PDEs have appeared with the emergence of high-speed computers with large-scale storage capacity. However, because of its ease of use, the FDM remains a valuable technique for solving these problems. Some of the advantages and disadvantages of the FDM are summarised in Table 1.1.

Table 1.1 Advantages and disadvantages of FDM.

Advantages	Disadvantages	References
Simple to use and implement	Difficulties in representing irregular boundaries could be solved by the GFDM.	Loredo et al. (2016) Muelas et al. (2019) Harish et al. (2021)
Converges faster and more accurate	The large-sparse linear system of equations and sophisticated algorithms are required, but they are relatively difficult to code.	Fadugba et al. (2012)

Interpolation is a method in numerical analysis that can be used to construct or estimate new data points using known (previous) data. Interpolation can also be defined as the process of finding a formula whose graph will pass through a set of points. There are many types of interpolation methods, such as piecewise constant interpolation, linear interpolation, polynomial interpolation, spline interpolation, interpolation via Gaussian process, rational interpolation, trigonometric interpolation, and multivariate interpolation.

The asymptotic interpolation method (AIM) is a method that can be used to estimate an unknown value as the sample size of a problem goes to infinity. This method uses different asymptotic functions, which have enough parameters to capture the behaviour of a problem, as in Table 2.1 (Vyaz'min et al., 2001). Table 1.2 shows the advantages and disadvantages of the AIM.

Table 1.2 Advantages and disadvantages of the AIM.

Advantages	Disadvantages	References
Highly accurate approximations are obtained in only a few iterations		Süleyman Cengizci (2017)
Give accurate predictions at infinity	Must know the behaviour of fluid	Yukalov et al. (2010)

Based on the advantages of the FDM and AIM, the combination of these two methods could solve different models of nonlinear PDEs that have an infinite domain and provide an accurate result.

1.2 Statement of the Problem

Many studies have been conducted in an attempt to solve nonlinear PDEs that have been derived from modelling real-world issues. Accuracy, consistency, and stability are the major concerns in problem-solving. Over time, many numerical methods have been devised and proven to be quite effective for solving problems in physics and engineering. The FDM is one of the numerical methods most researchers use to discretise the nonlinear equation. It has been proved to provide accurate results and converge faster in a finite domain.

The main problem in this research is to find an accurate result in an infinite domain that is close to asymptote, specifically for higher orders and higher degrees of nonlinear PDEs of fluid flow problems. Third-and fourth-grade non-Newtonian fluids are chosen due to their complicated mathematical formulation, which involves constitutive equations.

An effective numerical method for solving infinite domains in a higher degree of the nonlinear PDE is required. A good algorithm in a numerical method could efficiently solve many problems in less time.

This research will combine two methods, which are the FDM to discretize the nonlinear PDE and the interpolation method to estimate an unknown value. To show the infinite domain, three or more different lengths will be highlighted. A special asymptotic function that has parameters will be inserted into the system. More length will produce more data. Thus, the nonlinear least square curve fit will be used to find the best data fit for parameters.

Error analysis and validation will be conducted to ensure the hybrid method can be used for many problems that involve an infinite domain. A theory of stability will be conducted to ensure the problem is stable with the numerical scheme.

From the discussion, the problems are related to:

(a) How to handle the problems involving infinite domains?

(b) How can the hybrid method be generated or introduced for any problems related to MHD acceleration (constant or variable) flows for third- or fourth-grade fluid?

(c) To what extent can the hybrid method error be agreed upon to ensure the method is valuable and stable?

(d) What are the advantages of the hybrid method?

(e) What are the effects of the rotation parameter, magnetic parameter, third- and fourth-grade parameters, porous parameter, Prandtl number, and Grashof number on the velocity and temperature?

1.3 Objectives of the Study

The main purpose of this study is to introduce a hybrid approach that combines the FDM with the AIM for third- and fourth-grade fluids. Other objectives of this study are as follows:

(a) To determine the range of error and norm of error in error analysis, which includes perturbation systems,

(b) To obtain the theory of stability for numerical method problems,

(c) To obtain an approximate solution for the constant or variable acceleration of non-Newtonian fluid flow in a rotating frame for third- and fourth-grade fluids using the hybrid method,

(d) To analyse the effects of parameters on the velocity and temperature distribution.

1.4 Scope of the Study

Third- and fourth-grade non-Newtonian fluids with the constant or variable acceleration of unsteady MHD flow in a rotating frame and porous medium are studied. The presence of heat transfer in the fluid flow problem is also investigated. A new hybrid, FDM and AIM, are used to obtain approximate numerical solutions. This study has the following assumptions:

(a) The third- and fourth-grade fluids,

(b) The fluid flow problem is in an unsteady state that varies with time,

(c) The flow is in a rotating frame,

(d) The fluid conducts electricity,

(e) Heat transfer appears in fluid flow, and

(f) The fluid flow passes through a porous medium.

1.5 Significance of the Study

The study of fluid flow behaviour has grown in popularity as it exists in various technical and industrial domains, such as the manufacturing of plastic and food. The effects of MHD, rotation and heat transfer on fluid flow have inspired researchers to develop new machines like MHD generators and pumps. The complex system of non-Newtonian fluids challenges applied mathematicians with hurdles in developing suitable and available algorithms for fluid flow problems. Several fluid flow models have been proposed in response to novel fluid behaviour problems. Many fluid flow problems have been solved analytically to generate a formula for an exact solution or numerically to obtain an approximation of the true solution. The findings of this study are noteworthy in the following ways:

(a) Because of the existence of a semi-infinite fluid flow problem, the idea of developing a new hybrid numerical method has arisen,

(b) It is hoped that the hybrid approach will spur further research into various types of non-Newtonian fluid flow problems, and

(c) The approximate numerical solution obtained is valuable to determine the accuracy of the analytical solution.

1.6 Outline of the Thesis

This study is divided into five chapters. Chapter 1 is the introduction section that defines non-Newtonian fluids and discusses the differential fluid types, namely second-, third- and fourth-grades, MHD, rotating frame and porous medium. It also highlights previous research, particularly in terms of engineering and industrial applications. In addition, Chapter 1 outlines the study’s objectives. It is important to note that this study focuses on incompressible third- and fourth-grade fluids, constant and variable accelerations, MHD, time-dependent, rotation, porous medium, heat transfer and numerical methods.

Chapter 2 is an overview of previous research on non-Newtonian fluid flow problems by focusing on second-, third- and fourth-grade fluids, numerical FDM, AIM, nonlinear least square curve fitting and hybrid technique. This study aims to address the research gap left by previous research.

On the other hand, Chapter 3 explains in detail the research method. Figure 3.1 depicts the flow chart of the research methodology, which is further divided into the following sections: information gathering, algorithm evaluation (finite difference, asymptotic interpolation and hybrid generalisation methods), algorithm implementation and validation. The principles of the FDM and the algorithms for solving nonlinear equations are covered.

Furthermore, the concept of the AIM is described mathematically and graphically in Figure 3.3. Four steps are inferred in the hybrid method's generalisation section. The concept of the nonlinear least square method is also explained in Step 4. Besides, a MATLAB coding algorithm is included in this chapter to discuss the AIM. Figure 3.4 depicts the flow diagram for the novel hybrid technique approach. This part also covers validation and error analysis.

Four fluid problems are studied to validate the hybrid finite difference and asymptotic interpolation methods. The first problem is non-Newtonian third-grade fluid constant acceleration in a rotating frame. The governing equation is studied, and the hybrid numerical method is applied to the system. The results are then compared to those of prior studies using the HAM. Previously, there was no exact solution to this problem. Hence the numerical scheme of the modified third-grade fluid is constructed with an exact solution. Then, error analysis and stability tests are conducted. At the same time, the second validation is conducted by solving the problem of non-Newtonian third-grade fluid flow with variable accelerations in a rotating frame.

Next, validation with the MHD third-grade fluid flow in a rotating frame and porous medium has been conducted in validation 3. The fourth validation is conducted by solving nonlinear second-order partial differential equations in third-grade fluid flow at the rotating cylinder. After that, the results of the hybrid numerical solution are compared against the exact solution from previous research.

Moving on, Chapter 4 presents a new fluid flow problem related to the constant acceleration of unsteady third-grade MHD fluid in a rotating frame. The governing equations for this problem are presented. The problem is related to the unbounded BC. Therefore, this new approach to the hybrid method is applied. The results are validated by introducing an implicit numerical scheme of the modified third-grade fluid. An error analysis is used to calculate the exact difference and relative error between the hybrid approach and the exact solution. The analysis then continues by varying the parameter values to examine how fluid flow affects the velocity profile.

An investigation into the new fourth-grade fluid problem has been conducted, which solves the variable acceleration of the unsteady fourth-grade MHD fluid in the rotating frame. Compared to the third-grade fluid, the fourth-grade fluid's governing equations are more cumbersome and complex. Due to the time-dependent and complex equation appearing in this problem, the FDM is carefully done at this time $t = 1$ and $t = [2, N-1]$. Next, the AIM is inserted into the process to satisfy infinite length conditions. The investigation continues by varying the parameter values to see how they affect the fluid flow's velocity profile. This study continues to address the problem of heat transfer in fluid flow in MHD third-grade fluid flow in a rotating frame with and without the porous medium.

Finally, Chapter 5 discusses the conclusion and recommendation. The research findings include theoretical error analysis of the FDM and AIM, stability tests, the solution of three fluid flow problems for validation purposes, and four new fluid flow problems. The contributions of knowledge highlighted in this study are the introduction of a hybrid FD-AIM approach, the presentation of theoretical error analysis, the application of the θ-method and a new solution to the fluid flow problems.

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LIST OF PUBLICATIONS

1) JOURNAL/ PROCEEDING/ CONFERENCE:

Shafaruniza Mahadi, Yeak Su Hoe, Norazam Arbin & Faisal Salah (2021). Numerical Solution for Unsteady Acceleration MHD Third-Grade Fluid Flow in a Rotating Frame through Porous Medium Over Semi-Infinite Boundary Condition with a Presence of Heat Transfer. Journal of Advanced Research in Fluid Mechanics and Thermal Sciences 87, Issue 2, 90-105.

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5th International Conference on Mathematical Applications in Engineering (ICMAE' 2019). Conference date: 30-31 October 2019. Location: Putrajaya, Malaysia

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INTERNATIONAL CONFERENCE ON INFORMATICS, COMPUTING & APPLIED MATHEMATICS (ICICAM 2017). Conference date: 7-9 October 2017. Location: UNIVERSITI SULTAN ZAINAL ABIDIN (UNISZA), TERENGGANU

Mahadi, S., Yeak, S.H., Aziz, Z.A. & Salah, F. (2016). A New Numerical Solution of Hybrid Method for Third Grade Fluid Flow Problem. ISBN: 9789670479538. International Conference on Applied Computing, Mathematical Sciences and Engineering. Conference date: 30 - 31 May 2016. Location: Berjaya Waterfront Hotel, Johor Bahru, Johor Bahru, Malaysia.

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Conference date: 12–14 August 2014. Location: Langkawi, Kedah Malaysia

ISBN: 978-0-7354-1274-3