FIRST ORDER DIFFERENTIAL EQUATION WITH PIECEWISE CONSTANT ARGUMENT

MOHD SHAHRULLIZAM BIN AHMAD

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DEDICATION

To my beloved family, lecturers and my friends, thank you for your great support in term of physically and mentally.

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ABSTRACT

In recent years, differential equation with piecewise constant argument (DEPCA) have started to play crucial roles of the phenomena in the real world. There are many problems occurred involving piecewise constant arguments in the field of mathematics, science and engineering. However, behaviour of the system with piecewise constant arguments is more complex when compared to the corresponding continuous regular systems. This study focused on solving first order nonlinear differential equations with piecewise constant argument. There are two types of differential equation which are the first order differential equation and the Bernoulli-type equations. The mathematical approaches are based on the DEPCA of simple forms, especially in the form of $[t]$ to which a continuous system is considered directly over a unit interval. In order to solve these equations, the condition of existence and uniqueness are used to determine both the types of equations for some class functions of g. Thus, an explicit form of solutions for a certain class of Bernoulli-type equations was solved indirectly by using the previous approached. At the same time, the periodicity conditions for these solutions were established. Several examples were explained for both types of differential equation and some graphs were obtained by using MATHEMATICA software.

ABSTRAK

Kebelakangan ini, persamaan pembezaan dengan pemalar hujah cebis demi cebis mula memainkan peranan yang sangat penting dalam fenomena realiti kehidupan. Terdapat banyak masalah yang berlaku melibatkan cebisan hujah malar dalam bidang matematik, sains dan kejuruteraan. Bagaimanapun, tingkah laku sistem dengan cebisan hujah malar adalah lebih kompleks jika dibandingkan dengan sistem tetap yang berterusan. Kajian ini berfokus untuk menyelesaikan persamaan pembezaan bukan linear peringkat pertama dengan pemalar hujah cebis demi cebis. Terdapat dua jenis persamaan pembezaan, iaitu pembezaan peringkat pertama dan persamaan jenis Bernoulli. Pendekatan matematik adalah berdasarkan DEPCA berbentuk mudah, terutamanya dalam bentuk [t] dimana sistem berterusan dianggap secara langsung melalui selang unit. Untuk menyelesaikan persamaan ini, syarat kewujudan dan keunikan digunakan kepada kedua-dua persamaan untuk beberapa kelas fungsi g. Oleh itu, bentuk penyelesaian yang ekplisit untuk kelas persamaan jenis Bernoulli telah diselesaikan secara tidak langsung dengan menggunakan pendekatan yang terdahulu. Pada masa yang sama, syarat keberkalaan bagi penyelesaian ini telah dihasilkan. Beberapa contoh telah dibincangkan bagi kedua-dua jenis persamaan pembezaan dan beberapa graph diperolehi dengan menggunakan perisian MATHEMATICA.

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LIST OF ABBREVIATIONS

DEPCA	-	Differential Equation with Piecewise Constant Argument

LIST OF SYMBOLS

$\left[t\right]$	-	Greatest Function

CHAPTER 1

INTRODUCTION

1.1 Introduction

Differential equations with piecewise constant argument (DEPCA) are known for producing many mathematical models used to describe many phenomena in real life. In these equations, the derivatives of the unknown functions depend on not just the time *t* before *t*. Since this is essentially expressing the derivatives on terms of the solution at discrete points of time before *t*, it is usually referred to as a hybrid system, other examples of the application of these equations to the problems of biology, cellular neural networks and mechanical systems can be in [1-4].

DEPCA has been under intensive investigation for the last twenty years. The first studies in this field have been given in [5,6], after this, stability, contractivity and existence of periodic solutions. The general theory and basic results for DEPCA have been thoroughly investigated in [7]. Nowadays, much research has been focused on the numerical solutions of EPCA, including convergence[8], stability[9] and oscillation[10]. However, we can't find any results concerning the approximate analytical solution of DEPCA have been published.

In general, a system of piecewise constant argument is unique compared to corresponding continuous regular systems. The complete solutions of the DEPCA are usually based on the continuity at the point joining the neighboring intervals. Therefore, the solutions of the differential equations governing the DEPCA combine the features of both differential and difference equations. The overwhelming majority of the investigations in the field of the DEPCA are pure mathematical approaches.

1.2 Problem Background

Since the beginning of 1980, DEPCA have become a major attraction to researchers in mathematics, science and engineering. This field has long been extensively developed and expanded to areas such as biomedicine, chemistry, mechanical engineering, physics, civil engineering, and aerodynamical engineering. This phenomenon is closely related in the field of physics and engineering systems to stepwise or piecewise constant variables or motion under piecewise constant forces. In mathematical forms, it is used for first or second order differential equations or systems DEPCAs.

DEPCAs are encountered in biological phenomena [11,12], the stabilization of hybrid control systems with feedback discrete controller [13], or damped oscillators [14]. Later, several authors discussed some articles related to stability, oscillation properties, and the existence of periodic solutions [2, 7, 15].

In [16] the authors construct the Green's function related to the linear operator $x'(t)+mx(t)+Mx([t])$ from where they obtain some maximum principles in the space of periodic solutions depending on the values of the real parameters m and M. These operators were also studied with initial value conditions in [17]. In [18], the method of lower and upper solutions was used to derive from the existence of periodic solutions of a first order nonlinear DEPCA. This technique, as well as the technique of weakly coupled lower and upper solutions, is used in [19] to deduce the existence results of a first order nonlinear boundary value problem of involving a differential equation with continuous delay. The arguments merge the techniques used for equations in [19] with the continuous delay produced for linear first order piecewise equations in [16].

In [12], the following problem has been considered.

$$y'(t) = f(t, y[t-k]),\ t \neq n,\ t \in J,$$
$$\Delta y(n^{+}) = I_n(y(n)),\ n = 1, 2, \ldots, p,\ y(0) = y(T).$$

Using the method of upper and lower solutions, it is expressed in [20] that at least one solution is present. However, in [13] the existence and uniqueness of the initial value problem solutions have been proved

$$x'(t) = f(x(t), x(g(t))),\ x(0) = x_0,$$

and they also studied the oscillation and stability cases in which f is a continuous function and g: $[0, \infty) \rightarrow [0, \infty)$, $g(t) \leq t$ is a step function.

1.3 Problem Statement

There are many phenomena related to piecewise constant variations that are often seen in the real world. These phenomena may often be modeled by piecewise constant systems with corresponding to differential equations containing piecewise constant arguments. Accordingly, many studies have been conducted on DEPCAs, but the results for the first order nonlinear differential equations with piecewise constant are limited. In addition, the solutions for DEPCAs are unique and not easy to be analyzed or solved.

In this study, to solve first order differential equation with piecewise constant argument. There are two forms of differential equation with piecewise constant argument having the forms;

(a) $$x'(t) = x(t) \cdot g\left(x\left(\left[t\right]\right)\right) \tag{1.1}$$

(b) $$x'(t) = x^n(t) \cdot g\left(x\left(\left[t\right]\right)\right), \quad n \neq 1 \tag{1.2}$$

where g is continuous and $[t]$ is the greatest-integer function.

We define a solution of equation (1.1) and (1.2) as continuous function satisfying the equation within the intervals of the argument constancy. Then, we describe existence condition and solve equation. Lastly, we also give the properties of equations having only periodic solutions.

1.4 Research Objective

a) To give uniqueness condition for equations (1.1) and (1.2)

b) To solve the equations of (1.1) and (1.2) for some class functions *g.*

c) To define existence conditions of periodic solution of (1.1) and (1.2)

1.5 Scope of Study

The research will focus on first order nonlinear DEPCA governed by the differential in the form of equations (1.1) and (1.2). The mathematical approaches are based on the DEPCAs of simple forms, especially in the form of $[t]$ to which a continuous system is considered directly over a unit time interval. Also, the scope of the study will be focused on existence conditions equations (1.1), (1.2) and solving both equations for some class functions of g.

1.6 Significance of the Study

The outcome of the dissertation clearly explained the solution of first order differential equation and solution of Bernoulli type equation. Researchers can use the solution that has been developed to solve problems involving differential equation with piecewise constant argument. In addition, the examples produced also help the researcher to understand the concept of piecewise constant argument in more detail. Last but not least, this study will contribute to a new knowledge of theoretical in first order differential equations with piecewise constant arguments.

1.7 Research Methodology

The research begins by studying a brief reviews of DEPCAs from previous researchers and highlighted the importance and the limitations for studying the solutions of these equations. Next, we apply the existence and uniqueness condition developed by J.Wiener and V. Lakshmikantham [17] to solve the both types equations. Then, we also give the properties of equations having only periodic solutions. Lastly, some numerical solutions for both types equations are given. The research methodology is summarized in Figure 1.1.

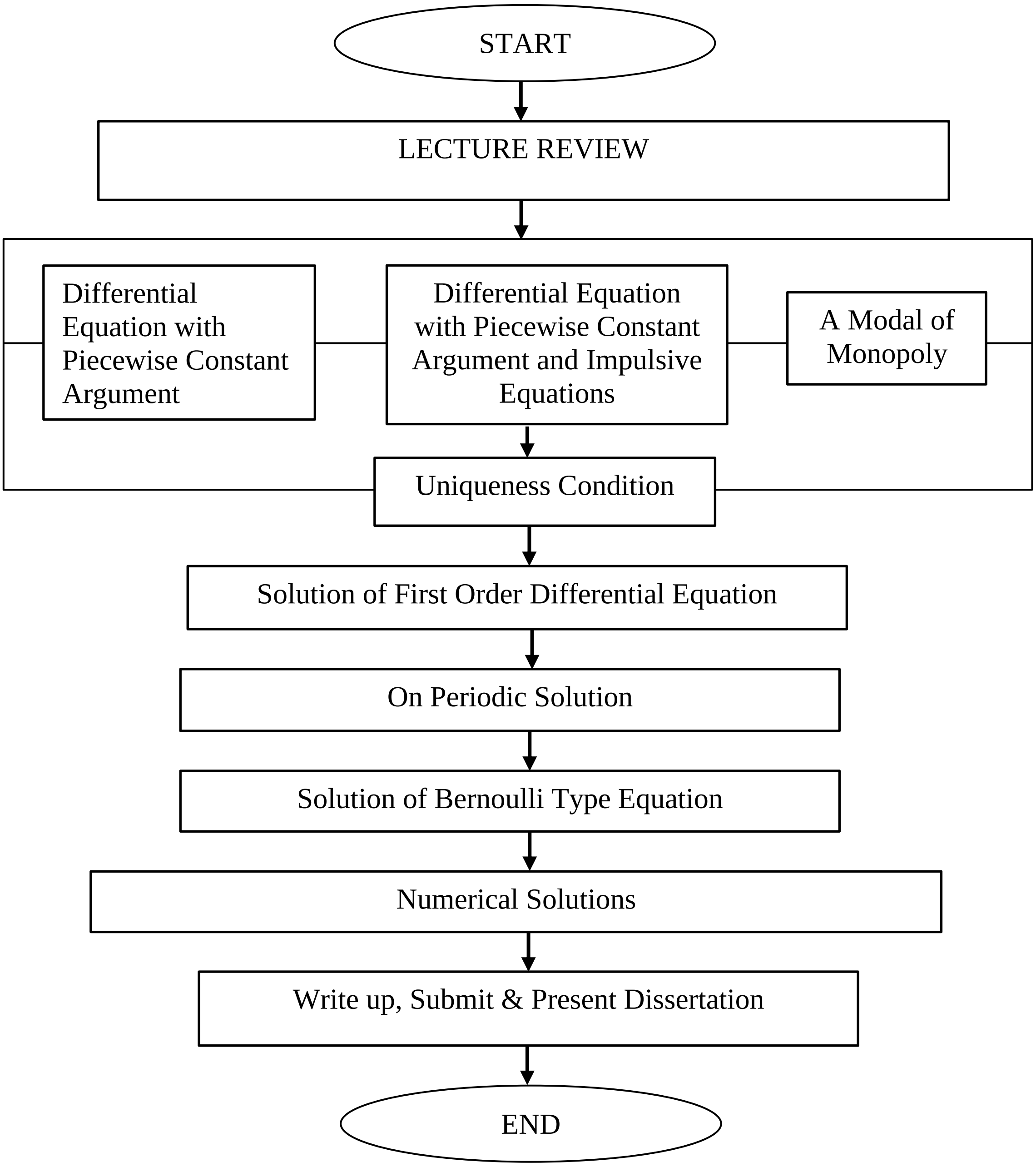

Figure 1.1 Research Methodology

1.8 Dissertation Organization

This dissertation is organized into five chapters. Chapter 1 contains the study framework. Background of the problem, statement of the problem, objectives, scope, and significance of the study are discussed in this chapter.

Chapter 2 discusses the literature review of this research. Different works on first order DEPCA by various researchers are explained in details.

In Chapter 3, the methodology will be discussed. This chapter introduces the sequence of the process in solving the nonlinear first order DEPCA in the form (1.1) and (1.2). The theorem and also lemma will be stated and explained in detail to extract the information in order to solve the equation. Lastly, the first objective of the project will be achieved in this chapter.

Chapter 4 shows the end result of both cases and discussions based on the solution obtained in Chapter 3.

Finally, the conclusion of the whole research is explained in Chapter 5. Also included in this chapter is future recommendation relating to the DEPCA.

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