PRIME ORDER AND COMPOSITE ORDER CAYLEY GRAPHS OF GENERALISED QUATERNION GROUP AND QUASI-DIHEDRAL GROUP

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A dissertation submitted in partial fulfilment of the
requirements for the award of the degree of
Master of Science

Faculty of Science
Universiti Teknologi Malaysia

MARCH 2020

To my family, with love.

ACKNOWLEDGEMENT

All Praise to ALLAH S.W.T the Almighty, for giving me the blessing, the strength, the chance and endurance to complete this study.

I would like to express my sincere appreciation to my supervisor Dr. Hazzirah Izzati Mat Hassim and my co-supervisor, Associate Professor Dr. Nor Muhainiah Mohd Ali for their time, generous guidance, patience and encouragement throughout the whole dissertation, from which I have learned a lot. Their guidance helped me in all the time of research and writing of this dissertation. I could not have imagined having a better advisor and mentor for my Master's studies.

I must express my very deepest gratitude to my family, especially my parents, for providing me with unfailing support and continuous encouragement throughout my years of study and through the process of researching and writing this dissertation. This accomplishment would not have been possible without them. Thank you very much.

Last but not the least, I would like to thank my classmates, lecturers, Applied Algebra and Analysis Group (AAAG) members, friends, and everyone for the kindness and love you have shown to me during my study years here.

ABSTRACT

A Cayley graph is a structure consisting of vertices and edges that describes the information of a group and its generators where two vertices are connected by a directed edge in certain conditions. This research focuses on the prime order and composite order Cayley graphs on the generalised quaternion group and the quasi-dihedral group, where the subsets are the set of prime order and composite order element of each group. The properties of elements of both groups are investigated, and then the structures of prime order and composite order Cayley graphs of generalised quaternion group and quasi-dihedral group are obtained. Besides, the properties of the graph such as chromatic number, independence number, clique number, diameter, girth, and graph planarity are found. From this research, it is shown that the prime order and composite order Cayley graphs of both generalised quaternion group and quasi-dihedral group consist of unions of isomorphic components of equal vertices and a regular connected graph, respectively.

ABSTRAK

Graf Cayley ialah struktur yang terdiri daripada bucu-bucu dan tepi-tepi yang menghuraikan maklumat sesebuah kumpulan dan penjananya di mana dua bucu berkait oleh tepi berarah dalam keadaan-keadaan tertentu. Kajian ini memberi tumpuan pada graf Cayley peringkat perdana dan peringkat gubahan pada kumpulan kuaternion teritlak dan kumpulan kuasi-dihedral, di mana subsetnya adalah set unsur peringkat perdana dan peringkat gubahan kumpulan tersebut. Sifat unsur-unsur kumpulan tersebut disiasat, dan kemudiannya struktur graf Cayley peringkat perdana dan peringkat gubahan kumpulan kuaternion teritlak dan kuasi-dihedral diperoleh. Di samping itu, sifat graf seperti nombor kromatik, nombor ketakbersandaran, nombor klik, diameter, lilitan dan kesatahan graf ditemui. Daripada kajian ini, ia menunjukkan bahawa graf Cayley peringkat perdana dan peringkat gubahan untuk kedua-dua kumpulan kuaternion teritlak dan kuasi-dihedral masing-masing terdiri daripada kesatuan komponen isomorfisma bucu sama dan graf sekata terkait.

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LIST OF SYMBOLS

Q_{2^n}	-	Generalised quaternion group of order 2^n
QD_{2^n}	-	Quasi-dihedral group of order 2^n
$\langle a \rangle$	-	Cyclic group generated by an element a
$\cup$	-	Union
$\subset$	-	Subset of
$\in$	-	Element of
e	-	The identity element
a^{-1}	-	The inverse of an element a
C_n	-	Cyclic group of order n
$\gcd(m, n)$	-	Greatest common divisor of integers m and n
$\mathbb{N}$	-	Set of all natural numbers
Sym_n	-	Symmetric group of order $n!$
S_i	-	Subset of a group
$\lvert a \rvert$	-	Order of an element a in a group G
φ	-	Euler totient function
K_n	-	Complete graph with n vertices
$K_{m,n}$	-	Complete bipartite graph with m and n vertices
$V(\Gamma)$	-	Vertex set of a graph Γ
$E(\Gamma)$	-	Edge set of a graph Γ
$deg(v)$	-	Degree of vertex v in a graph Γ
$Cay(G, S)$	-	Cayley graph associated to a group G with subset S in G
$Cay_p(G, S)$	-	Prime order Cayley graph associated to a group G with subset S in G
$Cay_c(G, S)$	-	Composite order Cayley graph associated to a group G with subset S in G
$\chi(\Gamma)$	-	Chromatic number of a graph Γ
$\alpha(\Gamma)$	-	Independence number of a graph Γ
$\omega(\Gamma)$	-	Clique number of a graph Γ

$\mathrm{diam}(\Gamma)$	-	Diameter of a graph Γ
$\mathrm{girth}(\Gamma)$	-	Girth of a graph Γ

CHAPTER 1

INTRODUCTION

1.1 Background of the Research

A graph is a structure consisting of vertices and edges such that there exists a relation on connecting these vertices with the edges. Generally, to construct a graph, the vertex and edges are drawn in the form of dots and lines.

A Cayley graph (or sometimes called Cayley digraph, from the word directed graph) is a graph that visualises the abstract structure of a group. The finite group is typically considered in this formation of the graph, such that the set of generators is used to define the presentation of the group. A Cayley graph is denoted by $Cay(G, S)$, where G is a finite group and S is a subset of G. This type of graph is studied extensively as a method of application in various fields, such as computer science [1], electrical engineering [2] and bioinformatics.

Studies on the Cayley graphs of groups have developed rapidly in the past few decades. Sabidussi is one of the researchers who started to use the terminology of Cayley graph from the view of vertex-transitive graph [3]. Since then, a lot of research on Cayley graph has been done by many authors [4, 5, 6].

Tolue introduced the prime order Cayley graph [7] and the composite order Cayley graph [8], a derivation of Cayley graph in relation to the subset of the graphs, which are the set of prime order and composite order elements, respectively. The author covered the construction and successfully generalised the properties of prime order and composite order Cayley graphs of cyclic groups and dihedral groups of any finite order.

1.2 Problem Statement

The introduction of prime order and composite order Cayley graphs is inspired by the importance of the order of the elements of a group. Although prime order and composite order Cayley graphs is a new topic in the graph associated to groups, there are numerous results obtained on the determination of structure and planarity of the graphs on several main classes of groups. However, previous studies of prime order and composite order Cayley graphs have not dealt with the generalised quaternion group and quasi-dihedral group, that is up to date, the structure of these graphs for both groups have not been acquired. The structures and the properties of the prime order and composite order Cayley graphs of these groups also have not been investigated before although certain properties such as chromatic numbers can be applied in various field, for example, in mobile network. Hence, this research is conducted to discover the structures of the generalised quaternion group and quasi-dihedral group in the form of prime order and composite order Cayley graphs. Furthermore, this research determined the properties of prime order and composite order Cayley graphs of both groups.

1.3 Research Objectives

The objectives of this research are listed as follows:

a) To identify the subsets of prime order and composite order element of the generalised quaternion group and quasi-dihedral group.
b) To construct prime order and composite order Cayley graphs of generalised quaternion group.
c) To construct prime order and composite order Cayley graphs of quasi-dihedral group.
d) To determine some properties of the graphs obtained in (b) and (c).

1.4 Scope of the Study

This research focuses on prime order and composite order Cayley graphs of the generalised quaternion group and quasi-dihedral group, precisely on the characteristics and the properties of the graphs including chromatic number, independence number, clique number, girth, diameter and planarity of the graphs.

1.5 Significance of Findings

The concept of graphs associated with groups has been an interesting topic for the last century with many findings being applied to various fields. The methodology of utilising the properties of the group to the graph has been widely used by researchers, especially on Cayley graph, such as interconnection network and design, campanology, word metric and many more. Hence, the output of this research will give advancement in both group theory and graph theory as it could help the researchers to understand more about the concept of graph associated to groups. Furthermore, the construction of prime order and composite order Cayley graphs may help some researchers to generate some fresh insight into the construction of these graphs on some other finite groups.

1.6 Research Methodology

In this study, the subset, S of the group G is determined from the condition of S, such that its element are of prime or composite order. Then, the graphs of the group is constructed based on the subset S. The properties of the graph formed are determined such as chromatic number, independence number, clique number, girth, and diameter, and graph planarity. The pattern of the graph is observed and analysed before determining a generalisation on the formation of the graph.

1.7 Dissertation Organisation

This dissertation is divided into six chapters. The first chapter provides an introduction to the whole dissertation, which inclusive of the background of the research, problem statement, research objective, scope, the significance of the research, and research methodology.

Chapter 2 consists of the literature review of this research. Basic properties of generalised quaternion group and quasi-dihedral group are described. Furthermore, the preliminaries on the graph and its properties are given. Also, the concept of the graph associated to groups is presented, including the definitions and previous research on prime order and composite order Cayley graphs.

Next, Chapter 3 presents the methodology of prime order and composite order Cayley graphs construction. The construction of prime order and composite order Cayley graphs are done on some finite groups, including the analysis of the properties of constructed graphs.

Chapter 4 shows the construction of prime order and composite order Cayley graphs on generalised quaternion group. After the constructions of the graphs are established, the structures of these graphs are investigated to obtain these graph properties. Similar work is done in Chapter 5 for the quasi-dihedral group.

Chapter 6 concludes the dissertation with a summary of this research and proposing some recommendations for further research.

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