STABILITY ANALYSIS OF LAMINATED BEAM SYSTEMS WITH DELAY USING LYAPUNOV FUNCTIONAL

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ABSTRACT

This work is concerned with systems of laminated beams model subject to linear and nonlinear delay feedback. In a dynamic laminated beam, time delay manifests in the form of lags in restoring the desired system stability after perturbations. Four prevalent categories of time delay are considered. For laminated beams with relatively high adhesive stiffness, a constant delay feedback is considered for systems made up of individual beams with same elasticity, and neutral delay otherwise. In systems where delay is significantly due to adhesive softening, distributed delay is considered. Lastly, in structures where the mechanism of dissipating energy is nonlinear, a corresponding nonlinear delay effect is investigated. The mechanism of stabilization mainly relies on the intrinsic structural damping, unlike in previous works where researchers introduced additional dampings such as boundary feedback and material damping. The objective of this work is to establish the asymptotic behavior of a vibrating Timoshenko laminated beam using structural or utmost a single frictional damping in presence of different forms of time delay. The energy method for partial differential equations is the main tool used to establish wellposedness results and asymptotic behavior. The existence and uniqueness of the solution is proved using the linear semi group theory, whereas for energy decay properties, the multiplier technique involving constructing a suitable Lyapunov functional equivalent to the energy is utilized. With appropriate assumptions on the delay weight and wave speeds, it is established that the energy of the system at least decays exponentially due to structural damping. Furthermore, a single additional frictional damping guarantees polynomial decay despite the presence of constant or distributed delay feedback. For nonlinear structural damping, with help of some convexity arguments, general decay result is achieved. In summary, depending on the damping mechanism(s), exponential, polynomial, or general decay results of a laminated beam system subject to different forms of delay feedback are established.

ABSTRAK

Kerja ini berkaitan dengan sistem model rasuk berlamina tertakluk kepada maklum balas lengah linear dan tak linear. Dalam rasuk berlamina dinamik, lengah masa nyata dalam bentuk susulan bagi memulihkan kestabilan sistem yang diingini selepas gangguan. Empat kategori lazim lengah masa dipertimbangkan. Untuk rasuk berlamina dengan kekukuhan perekat yang agak tinggi, maklum balas lengah berterusan dipertimbangkan untuk sistem yang terdiri daripada rasuk individu dengan keanjalan yang sama, dan lengah neutral sebaliknya. Dalam sistem di mana lengah ketara disebabkan oleh pelembutan perekat, lengah teragih dipertimbangkan. Akhir sekali, dalam struktur di mana mekanisme pelesapan tenaga adalah tak linear, kesan lengah tak linear yang sepadan disiasat. Mekanisme penstabilan terutamanya bergantung pada redaman struktur intrinsik, tidak seperti dalam kajian sebelumnya di mana penyelidik memperkenalkan redaman tambahan seperti maklum balas sempadan dan redaman bahan. Objektif penyelidikan ini adalah untuk mewujudkan kelakuan asimptot rasuk berlamina Timoshenko yang bergetar menggunakan struktur atau sepenuhnya redaman geseran tunggal dengan kehadiran pelbagai bentuk lengah masa. Kaedah tenaga untuk persamaan pembezaan separa ialah alat utama yang digunakan untuk mewujudkan keputusan sangat teraju rapi dan kelakuan asimptot. Kewujudan dan keunikan penyelesaian dibuktikan dengan menggunakan teori kumpulan separa linear, manakala bagi sifat pereputan tenaga, teknik pengganda yang melibatkan pembinaan fungsi Lyapunov yang sesuai bersamaan dengan tenaga digunakan. Dengan andaian yang sesuai mengenai berat lengah dan kelajuan gelombang, adalah terbukti bahawa tenaga sistem sekurang-kurangnya menyusut secara eksponen disebabkan oleh redaman struktur. Tambahan pula, satu redaman geseran tambahan menjamin penyusutan polinomial walaupun terdapat maklum balas lengah yang berterusan atau teragih. Untuk redaman struktur tak linear, dengan bantuan beberapa pembolehubah cembung, hasil susutan umum dicapai. Secara ringkasnya, bergantung pada mekanisme redaman, hasil eksponen, polinomial atau hasil susutan umum bagi sistem rasuk berlamina tertakluk kepada bentuk maklum balas lengah yang berbeza diwujudkan.

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LIST OF ABBREVIATIONS

GLT	-	Glue-laminated timber
PVB	-	Polyvinyl butyral
DDE	-	Delay differential equation
PDE	-	Partial differential equation
ODE	-	Ordinary differential equation

LIST OF SYMBOLS

ρ	-	Mass density per unit length
A	-	Cross sectional area of the beam
E	-	Elasticity modulus
I	-	Second moment of area of the beam's cross section
I_ρ	-	Cross section polar moment of inertia
K	-	Shear modulus
l	-	Length
$q(x)$	-	Load per unit length of the beam
G	-	Shear stifness
D	-	Flexural rigidity
γ	-	Adhesive stiffness
β	-	Adhesive damping coffecient
μ_2	-	Delay weight
τ	-	Time delay
μ_1	-	Frictional damping coffecient
L	-	Linear form
B	-	Bilinear form
$\mathbb{F}$	-	Field
$\mathcal{H}$	-	Hilbert space
$\mathcal{B}$	-	Banach space
$\|\cdot\|$	-	Norm
L^p	-	L^p space
$W^{m,p}$	-	Sobolev space
$\langle\cdot,\cdot\rangle$	-	Inner product
$\mathcal{L}$	-	Lyapunov functional
Ω	-	Domain

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CHAPTER 1

INTRODUCTION

1.1 Introduction

Most structures in mechanical, electrical, civil and aerospace engineering are made of single beam or several beams or plates. In building most of these structures, more than one beam or plate are bonded or stuck together by different means depending on the intended purpose of the structure. This implies, slipping or movement between plates is very likely to occur. In most cases purposely allowed but only up to certain limits, as it might result in structural damage or complete breakdown. Technically speaking, allowance of slip creates some damping which assists in restoring the equilibrium state of the system. Applications of laminated beams in structural engineering include glued–laminated timber (GLT) (Uzelac Glavinić *et al.*, 2020) that are used in construction of bridges, building roofs, furniture fabrications, wall paneling, and other wooden structures. PVB–laminated glass component (Pelayo *et al.*, 2017; Schulze *et al.*, 2012), on the other hand, are used in building car windscreens, glass railings, glass floors, skylight roofs, sunspaces, solar panel components, smart gadgets gorilla glass panels, etc. Another application is the fibre–metal laminated structures (Laliberte *et al.*, 2000; Mukesh and Hynes, 2019), which are applied mostly in components where more than one material metal is required. For example, dry irons, hulls of submarines, auto mobile bodies, truck leaf springs, etc.Maintaining the system at equilibrium state is paramount because in any control system, controlled output is always desired. It is therefore central that, in physical and engineering applications, the planned and designed frameworks are stable. This necessitates investigating and establishing the rate at which these systems can regain stability in case of any perturbations. Owing to the considerable importance of beam structures in the field of engineering, engineers and mathematicians often develop models describing vibrations of these beam structures to fully study their stability and other structural properties.

1.2 Background of the Problem

In this section, the background beam theories related to this work is presented.

1.2.1 Single beam models

Researchers acknowledged that the bending effect is the single most crucial element in a transversely vibrating beam at a very early stage. Thus, the evolution of beam theories that lead to solutions involving transverse displacement. There are mainly three single beam theories in this regard. These theories are the Euler-Bernoulli, Rayleigh, and Timoshenko.

Introduced around 1750 (Euler, 1960), the Euler-Bernoulli beam theory is the most elementary beam theory. It basically estimates deflection properties of a beam as loads are applied to it. The kinetic energy is due to lateral displacement and the energy is a result of the bending. It was derived on the assumptions that rotational displacements and shear deformations are negligible. Therefore, Euler-Bernoulli theory overestimates the natural frequencies especially in non-slender beams (Han *et al.*, 1999).

The Rayleigh beam theory (Rayleigh, 1896) incorporates the effect of rotation of the cross-section. As a result, it partially corrects the Euler-Bernoulli model's overestimation of natural frequencies. However, the natural frequencies are still slightly overstated because shear deformation is neglected (Davies, 1937).

Timoshenko (1921) introduced a beam model including both shear deformation and rotational bending effects. This takes care of Euler-Bernoulli beam theory assumptions. The Timoshenko beam theory is a significant improvement especially for non-slender beams and high-frequency responses where shear or rotary effects can not be ignored.

The following are the common assumptions made by Euler-Bernoulli, Rayleigh, and Timoshenko beam models:

- One dimension is significantly larger compared to the other two.
- Hookean material is used.
- The Poisson effect is negligible.
- The cross-sectional area is symmetric such that the neutral and centroidal axes coincide.
- Planes perpendicular to the neutral axis remain perpendicular after deformation.
- The rotation angle is relatively small so that the small angle assumption can be utilized.

1.2.2 Laminated beams

It is worth mentioning that, the classical single Timoshenko beam model lacks internal or external damping. Thus stabilization always depends on the additional damping mechanisms introduced into the system. On the other hand, for composite layered structures, it has been established that the interface connection property possesses a significant impact on the deformation as well as stress in the form of internal structural damping, depending on the material used in the structure (Ecsedi and Baksa, 2011; Wu *et al.*, 2016). This makes laminated beam structures more preferred in application. In structural engineering, adhesives are among the most used type of connectors of these layered beam structures. To this effect, Hansen and Spies (1997) introduced a differential model describing the vibrations in a structure set up by a pair of equal rods with uniform thickness, conjoined with help of an adhesive in a manner that interfacial slip is possible when in continuous contact with each other. The layer of adhesive at the interface is presumed to be of negligible mass and thickness, and produces a restorative frictional force proportionate to the amount of slip. This is frictional for referred to as structure damping. If the adhesive stiffness goes to infinity thus leading a no slip along the interface between the beams, then the system behaves like a single beam. On the other hand, if adhesive stiffness tends to zero, the two layers become separated. Hence the desired case is between the two extremes. Single and laminated beams theories and their properties are summarized in Table 1.1.

Table 1.1 Summary of single and laminated beam theories

Beam Model	Bending moment	Lateral displacement	Rotational bending	Shear deformation	Structural damping
Euler-Bernoulli	✓	✓	×	×	×
Rayleigh	✓	✓	✓	×	×
Timoshenko	✓	✓	✓	✓	×
Laminated	✓	✓	✓	✓	✓

The laminated beam model has attracted the attention of researchers due to it's applicability structural engineering. Like in any other model resulting from control system, establishing it's well posedness and the stability properties is paramount. To this effect, various damping mechanisms including frictional, thermoelastic, viscoelastic damping as well as damping by boundary controls have been used by mathematicians and engineers to achieve the desired stability results of the vibrations of the structure. Furthermore, researchers have investigated and established the existence, uniqueness, and smoothness properties of solutions of the laminated beams model under these stabilization mechanisms. Some of the most interesting results in literature regarding laminated beams model are presented and discussed in Chapter 2.

To attain stability, a vibrating laminated beam structure dissipates energy due to damping. However, in applications, laminated beam structures are subjected to external factors such as radiation, heat, moisture, among others, resulting into gradual degeneration over time. It may be in form of adhesive softening, wear and tear on the individual beams, and others. This translates into weak vibrations hence time lags in restoration of the system's stability. As mentioned in the previous paragraph, the asymptotic behavior of laminated beams has been satisfactory studied, however a lot more is still desired regarding effect of delay on energy decay of the vibration of these systems. Authors have only been able to establish stability results using external feedback control or dissipation through thermal damping factors in addition to adhesive damping. This limits the application of this model to systems with boundary controls similar to what has been already studied or where the materials used are significantly thermoelastic.

1.3 Delay Differential Equations

Many physical, economic, and engineering processes are spontaneous and often exhibit a gradual nature. In such applications, the spontaneous rate of change is affected by both the current state as well as the previous conditions. Hence, to exhaustively comprehend and investigate these processes, differential models that reflect both present and past occurrences are considered to be the best representation of such phenomena. Concisely, delay differential equations (DDEs) are differential models in which spontaneous change at any instant varies with the solution and perhaps its derivatives at preceding occurrences. Differential equations with time delay are also often referred to as functional differential equations, hereditary differential models, differential models with diverging argument, aftereffect differential systems, among other classifications.

Functional differential equations have been intensely investigated for two decades and still counting (Schmidt, 1911). In mid 20th century, the study of delay differential equations (DDEs) further developed significantly, mainly in the Soviet Union, as a consequence of its relevant manifestation in engineering models as a result the of use of automatic control systems. It is clear that most engineers had an insight about hereditary effects occurring in physical systems. However, due to insufficient theory to discuss such models in details, delay effect was often ignored. Hereditary effects in system automatically arises from the time lags between detection and responding to the information. The nature of the time lag detects the category of time delay. In the mid 20th century, Minorsky (1941) reached a milestone. He analyzed and represented controlled motion of a vessel with a sailing ballast using a practical differential equation, reflecting and expressing the time needed to correct the position of the ballast as a time lag. He further established that the sail vessel follows an oscillatory motion, provided the time delay is significantly large. It has been long established that delay effects on stability of controlled system can be devastating. The presence of a small delay may lead to a remarkably worsened performance in fully automated control systems, or may completely turn an initially stable system into a chaotic one. However, owing to the fact that time delays naturally occur in controlled systems, for reliable out put, it is important that they are well thought about and reflected in the blueprints while designing control feedback for these systems. During the 1950s, researchers developed more interest in the subject matter, which led to vital publications and fundamental texts (Bebernes, 1968; Bellman and Cooke, 1963; Krasovskii, 1959; Myshkis, 1951). In addition, the theory of time delay differential equations has also found its applications in diverse fields including demographic dynamics (Kuang, 1993), economics (Keller, 2010), biological systems (Israel, 2005; MacDonald, 2008), life sciences (Smith, 2011), modeling neural networks (Beuter *et al.*, 1993), mechanical systems controlled by feedback (Hu *et al.*, 2003), and (Kyrychko and Hogan, 2010) for other engineering applications. Richard (2003) cited some other interesting and worth mentioning phenomena in which time delays are explicitly reflected. A comprehensive literature and past study about delay effects on stability related to this work is given in Chapter 2.

In general, a straightforward functional differential equation in $y(t) \in \mathbb{R}^n$ is given by

$$\frac{d}{dt}y(t) = g\left(t, y(t), y(t-\tau_i)\right)_{i=1}^{n}$$

where τ_i are positive constants representing time lags such that $t \geqslant \tau_i$; $i = 1, \cdots, n$, and the functional operator $g : \mathbb{R} \times \mathbb{R}^n \times C^1(\mathbb{R}, \mathbb{R}^n) \longrightarrow \mathbb{R}$. If the time lags dependent on time, then $\tau_i = \tau_i(t, y(t))$. The delay term in delay differential equation (DDEs) manifest itself in many ways. Below are some examples of single dimension DDEs.

1. Constant delay

$$y'(t) = g(t, y(t), y(t-\tau)),$$

2. Distributed/continuous delay

$$y'(t) = g\left(t, y(t), \int_{\tau_1}^{\tau_2} \mu(r) y(t-r) dr\right)$$

3. Time varying delay

$$y'(t) = g(t, y(t), y(t-\tau(t))),$$

4. Neutral delay

$$y'(t) = g\left(t, y(t), y'(t-\tau)\right),$$

where $\tau > 0$ is the time delay (Apalara, 2013).

1.4 Problem statement

Like in any other vibrating structure, time delays in the Timoshenko laminated beam manifest in form of lags in attaining or restoring the desired system stability after perturbations due to internal or external factors, among others. In the single Timoshenko beam theory, the amplitude of the vibrations of the complementary displacements (transverse and angular) vanishes due to damping. A time delay translates into a forward phase shift which increases early time response, resulting into frequency dispersion in displacements (Moyer and Miraglia, 2014). This often requires stronger damping to counteract the longer time needed for decay. This delay effect is intrinsic in the Timoshenko laminated beam model as it is derived on assumption of Timoshenko theory. Structural damping in a laminated beam provides some dissipation, and on the assumption of equal wave speeds, it is sufficient for exponential stability in the absence of delay (Apalara, 2021; Apalara *et al.*, 2020a), see section 2.2. It is yet to be established if the internal structural damping can still solely stabilize the system in the presence of delay, as authors have so far chosen other damping mechanisms especially material damping and boundary feedback. It is yet to be established if the internal structural damping can still solely stabilize the system in the presence of delay. Authors have so far chosen other damping mechanisms especially material damping and boundary feedback. This work intends to prove that, under some conditions, the intrinsic structural damping due to interfacial slip in a laminated beam system subject to different forms of delay is sufficient for energy decay. Moreover, if the dissipation through structural damping is coupled with a single linear frictional damping, then it is expected that the system stabilizes polynomially.

Even though in most physical application, researchers often assume constant or discrete time delay representation, perhaps because of its ease to handle, estimate, and interpret physically, this is not always the case. Time delay manifests in different realistic representations depending on the underlying cause and other factors influencing it (Kyrychko and Hogan, 2010). Furthermore, it also significantly varies form system to system mainly depending on the physical properties. The aim of this work is to prove the asymptotic behavior of a Timoshenko laminated beam model subject to the four most prevalent forms of time delay feedback. These forms delay feedback are very common in control systems and other applications involving differential equations. For vibrating Timoshenko laminated beam, we explain the physical scenario under which each form of time delay may manifest, i.e. why the chosen representation is the most realistic and appropriate among others in the prevailing circumstances.

For laminated beams with relatively high adhesive stiffness, and the system behaves more like a single beam system (Hansen and Spies, 1997), in which transverse vibrations are more notable compared to vibrations in other displacements. To this effect, a a simple or constant time delay feedback in transverse displacement is considered for systems made up of individual beams with same elasticity and neutral delay (Kyrychko and Hogan, 2010) otherwise. In laminated beam systems where delay is significantly due to adhesive softening, because of viscoelastic property of adhesives (Groth, 1990), time delay which incorporates memory over a specified time interval is the most realistic representation, thus distributed delay (Nicaise *et al.*, 2008) in the effective rotation angle is considered. Lastly, in laminated beam structures where the mechanism of dissipating energy is nonlinear (Al-Hababi *et al.*, 2020; Elliott *et al.*, 2015), for general energy decay results, a corresponding nonlinear delay effect in structural damping is considered. For further reading about the physical motivation of the above types of delay and other fascinating delay applications in modeling practical problems, the reader may see (Hale and Lunel, 2013; Kolmanovskii and Myshkis, 2013) and references therein.

1.5 Objectives of the Study

In this work, the delay effect acting on complementary displacements and in the dynamics of slip on stability of a vibrating laminated beam structure is considered.

In absence of time delay, vibrations in the Timoshenko laminated beam exponentially decay due structural damping (Apalara, 2021; Apalara *et al.*, 2020a), or due to a single linear frictional damping acting through the effective rotation angle (Apalara *et al.*, 2020b). In presence of time delay, is this dissipation through structural or frictional damping still sufficient for energy decay of the solution of Timoshenko laminated beam? If possible, under what conditions on the system's parameters and delay weight? The aim of this work is to answer these questions affirmatively. To acheive our goal, an internally dissipative delayed Timoshenko laminated beam system without any form of material damping due to thermal or viscosity effect is investigated. In other words, the concern is a self stabilizing laminated beam system on only internal structural damping (linear and nonlinear) (Apalara, 2021; Apalara *et al.*, 2020a), or on a single linear frictional damping (Apalara *et al.*, 2020b), with an aim of establishing asymptotic behavior in presence of constant, distributed, neutral and nonlinear delay. The major objectives to accomplish this goal are as follows:

1. To find the total energy of the system and it's corresponding derivative.
2. To establish the delay weight condition in relation to damping coefficient or other system parameters.
3. To construct a suitable Lyapunov functional that is equivalent to the energy of the system.
4. To state and prove the energy decay results of the system depending on the underlying assumptions on wave speeds.

1.6 Significance of the Study

Laminated beam structures are highly applicable especially in structural engineering. Models describing vibrations in these structures have been studied extensively with an intent of establishing their stability by analyzing the underlying factors that might lead to instability, i.e. failure or delayed decay of the vibrations. In most control systems, time lags are normally diagnosed as a source of instability or performance deterioration. In a single Timoshenko beam, all forms of time delay are seen to distort stability. Therefore in applications, any form of delay that may occur a Timoshenko beam should be put into consideration. The classical laminated beam system is derived mainly on the assumption of the single Timoshenko beam theory, it is thus important that all forms of delay effect to laminated beam system are studied in a satisfactory manner. Moreover, there is no way one can comprehensively study such a model resulting from practical applications without considering delay effects as they are inherent in these processes. In the previous work, stabilization of delayed laminated beam system is based mainly on assumption that the materials used are either thermoelastic or viscoelastic enough to create the additional damping to aid energy decay. In this work, we ignore these two assumptions. We consider an internally dissipative laminated beam system (more general) without any form of material damping due to thermal or viscosity effect, and establish stability on a single intrinsic structural or frictional damping. The results in this work improve on the general applicability of the Timoshenko laminated beam model.

1.7 Scope of the Study

A classical laminated beam model with delay is considered. The following is the scope of the proposed study.

1. Form of delay and point of action:
   - (a) Constant delay in the transverse displacement.
   - (b) Distributed delay in the effective rotation angle.
   - (c) Neutral delay in the transverse displacement.
   - (d) Nonlinear delay in the third equation.
2. Damping mechanism:
   - (a) Internal linear frictional and/or structural damping for constant, distributed and neutral delay problems.
   - (b) Nonlinear structural damping for nonlinear delay problem.
3. Boundary data: Mixed Dirichlet–Neumann boundary conditions.

4. Methodology:

   (a) Well-posedness: Standard linear semigroup theory.

   (b) Stability: Multiplier technique.

1.8 Organization of the thesis

The rest of the thesis is organized as follows: In Chapter 2, the literature about stabilization of laminated beams without delay using different mechanisms is given. Secondly, some review about delay differential equations related to this work is presented. The chapter is wound up by summarizing what has been done so far regarding stability analysis of laminated beam systems with delay, and finally presenting research questions as well as gaps. Chapter 3 is concerned with the methodology which is mainly the energy method, i.e. the linear semi group for wellposedness and the multiplier technique for establishing stability results. In Chapters 5−7, asymptotic behavior of laminated beam system with constant, distributed, neutral, and nonlinear delay are respectively established using structural or frictional damping. The summary and conclusion of the overall study are given in Chapter 8.

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LIST OF PUBLICATIONS

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1. Mpungu, K., Apalara, T.A. and Muminov, M.: *On the Stabilization of Laminated Beams with Delay.* APPLICATIONS OF MATHEMATICS, **66**(5), 789–812, (2021). `https://doi.org/10.21136/AM.2021.0056-20`.

2. Mpungu, K. and Apalara, T.A.: *Stability result of Laminated beam with internal distributed delay.* JOURNAL OF MATHEMATICAL INEQUALITIES, **15**(3), 1075–1091, (2021). `http://doi.org/10.7153/jmi-2021-15-73`.

3. Mpungu, K and Apalara, T.A.: *Exponential Stability of Laminated beam with constant delay feedback.* MATHEMATICAL MODELLING AND ANALYSIS, **26**(4), 566–581, (2021). `https://doi.org/10.3846/mma.2021.13759`.

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5. Mpungu, K. and Apalara, T.A.: *Asymptotic behavior of a laminated beam with nonlinear delay and nonlinear structural damping.* HACETTEPE JOURNAL OF MATHEMATICS AND STATISTICS, (2022). `https://doi.org/10.15672/hujms.947131`.