PERTURBATION METHOD ON THE MODIFIED RAYLEIGH-PLESSET EQUATION IN PSEUDO-COMPRESSIBLE BUBBLY VISCOELASTIC LIQUID

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DEDICATION

To my beloved mother, wife and children

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ABSTRACT

In bubbly-liquid flow, a transient phenomenon is happened due to the dynamics of the bubble, pressure, or temperature at any location in the flow. It is well known that shock wave propagation in liquid media is strongly affected by the presence of bubbles that interact with the shock wave, and the effects of the gas bubbles. The presence of interfacial interactions between the bubble and the liquid, bubbles interaction and compressibility of the viscoelastic liquid flow inevitably make the problem a difficult one. Due to these factors that a mathematical model of bubbly-liquid flow becomes more complex than the transient flow encountered in single-phase flow. The mathematical model of a transient pseudo-compressible two-phase gas bubble in viscoelastic liquid flow and heat transfer is discussed. Specific models are derived to describe the shock wave propagation behaviour of the bubbly-liquid flow. The gas behaviour inside a spherical bubble under the shock wave is analysed using the polytropic models. The modified Rayleigh–Plesset equation that described the bubble dynamics in a pseudo-compressible viscoelastic liquid is derived using the idea of conservation of kinetic energy equation, incorporating the effect of a bubble to bubble interaction. Kelvin-Voigt (linear viscoelastic) liquid and second grade (nonlinear viscoelastic) liquid are the two specific liquids considered. The governing equations are approximately solved using reduction perturbation method, and then Korteweg-de-Vries-Burger (KdVB) equations are derived. Adomian decomposition method is applied to solve the equations numerically. The result shows that the combination of acoustic, thermal, and viscous damping effect causes rapid damping in the wave propagation in the bubbly viscoelastic liquid flow. For Kelvin-Voigt liquid, the size of the bubble radius has no influence on the amplitude of the shock wave from zero up to 40s, while for second-grade liquid, the bigger the bubble, the larger the amplitude of the wave propagation at $\tau = 8$s. For the thermal variation, for both Kelvin-Voigt and second grade liquids, it is observed that a higher polytropic index gives rise to the lower amplitude of the shock wave. This implies that an isothermal process will give rise to a shock with the highest amplitude and dissipate faster. A highly viscous Kelvin-Voigt liquid dissipates faster, while for second-grade liquid, the result indicates no effect in either the amplitude or the steepness on the shock wave. It is observed in the case of Kelvin-Voigt liquid, that there is a significant effect of the modulus of elasticity on the shock wave, and no difference in the amplitude of the shock wave as a result of variation of relaxation parameter for second-grade liquid. For both Kelvin-Voigt and second-grade liquids, the variation in the number of bubbles and cluster size has relatively no effect on the shock propagation in the parameter values under consideration. Global and local stability analysis of the KdVB equations are carried out, and many novel wave solutions to the equations are derived. This study is useful for the performance analyses of bubbly viscoelastic liquid flow in predicting shock propagation in the liquid, and provides useful information for the effects of heat transfer phenomenon and applied pressure on transient pseudo-compressible bubbly-liquid flow.

ABSTRAK

Dalam aliran cecair berbuih, fenomena fana terjadi disebabkan dinamik gelembung, tekanan atau suhu di sebarang lokasi dalam aliran. Telah diketahui bahawa perambatan gelombang kejutan dalam medium cecair sangat terkesan dengan kehadiran gelembung yang berinteraksi dengan gelombang kejutan, dan kesan daripada gelembung gas tersebut. Kehadiran interaksi antara muka antara gelembung dan cecair, interaksi gelembung dan kebolehmampatan aliran cecair viskoelastik pasti menjadikan masalah sangat sukar. Disebabkan oleh faktor-faktor ini, model matematik aliran cecair berbuih menjadi lebih rumit berbanding aliran fana yang ditemui dalam aliran fasa tunggal. Model matematik bolehmampat pseudo fana gelembung gas fasa-dua dalam aliran cecair viskoelastik dan pemindahan haba dibincangkan. Model khusus diterbitkan untuk menerangkan kelakuan perambatan gelombang kejutan dalam aliran cecair berbuih. Kelakuan gas di dalam satu gelembung sfera semasa gelombang kejutan dianalisa menggunakan model politropik. Persamaan Rayleigh-Plesset terubahsuai yang menerangkan dinamik gelembung dalam cecair viskoelastik bolehmampat pseudo diterbit menggunakan idea persamaan keabadian tenaga kinetik berserta kesan interaksi gelembung dengan gelembung. Cecair Kelvin-Voigt (viskoelastik linear) dan cecair gred kedua (viskoelastik tak linear) adalah dua cecair khusus yang dipertimbangkan. Persamaan menakluk diselesai secara hampir menggunakan kaedah usikan terturun dan seterusnya persamaan Korteweg-de-Vries-Burger (KdVB) diterbitkan. Kaedah penguraian Adomian digunakan untuk menyelesaikan persamaan tersebut secara berangka. Keputusan menunjukkan bahawa gabungan akustik, terma, dan kesan redaman likat menyebabkan redaman cepat semasa perambatan gelombang di dalam aliran cecair viskoelastik berbuih. Bagi cecair Kelvin-Voigt, saiz jejari gelembung tidak mempengaruhi amplitud gelombang kejutan dari sifar sehingga 40s, manakala bagi cecair gred kedua, semakin besar gelembung, makin besarlah amplitud perambatan gelombang berlaku apabila $\tau = 8$s. Bagi variasi terma, kedua-dua cecair Kelvin-Voigt dan gred kedua, diperhatikan bahawa indeks politropik yang lebih tinggi menimbulkan amplitud gelombang kejutan yang lebih rendah. Ini mengimplikasikan bahawa proses isoterma akan menimbulkan gelombang kejutan dengan amplitud tertinggi dan lesap dengan lebih cepat. Cecair Kelvin-Voigt likat yang tinggi lesap lebih cepat, manakala bagi cecair gred kedua, keputusan menunjukkan tiada kesan sama ada terhadap amplitud ataupun kecuraman pada gelombang kejutan. Diperhatikan dalam kes cecair Kelvin-Voigt terdapat kesan ketara terhadap modul keanjalan pada gelombang kejutan, dan tiada perbezaan pada amplitud gelombang kejutan sepertimana hasil daripada keputusan variasi pengenduran parameter bagi cecair gred kedua. Bagi kedua-dua cecair Kelvin-Voigt dan gred kedua, variasi bilangan gelembung dan saiz kluster secara relatifnya tidak memberi kesan terhadap perambatan gelombang kejutan bagi nilai parameter yang dipertimbangkan. Analisis kestabilan global dan tempatan bagi persamaan KdVB dilakukan dan banyak penyelesaian gelombang tersohor persamaan ini diterbitkan. Kajian ini berguna untuk menganalisa keupayaan aliran cecair viskoelastik berbuih dalam meramalkan perambatan gelombang kejutan dalam cecair dan menyediakan maklumat berguna bagi kesan fenomena pemindahan haba dan mengenakan tekanan ke atas aliran cecair berbuih bolehmampat pseudo fana.

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LIST OF ABBREVIATIONS

(2+1) KDV	-	Two Dimensional Korteweg-de-Vries Equation
(2+1) KDVB	-	Two Dimensional Korteweg-de-Vries-Burgers Equation
(3+1) KDV	-	Three Dimensional Korteweg-de-Vries Equation
(3+1) KDVB	-	Three Dimensional Korteweg-de-Vries-Burgers Equation
ADM	-	Adomian Decomposition Method
IVP	-	Initial Value Problem
KDV	-	Korteweg-de-Vries Equation
KDVB	-	Korteweg-de-Vries-Burgers Equation
NSE	-	Nonlinear Schordinger Equation
ODE	-	Ordinary Differential Equation
PDE	-	Partial Differential Equation
RMC	-	Research Management Centre
RP	-	Rayleigh–Plesset Equation
UTM	-	Universiti Teknologi Malaysia

LIST OF SYMBOLS

$\mathbf{A_1}, \mathbf{A_2}$	-	Rivlin–Ericksen tensors
B_n	-	Adomian polynomial
b	-	Resistive constant
$\mathbf{b}$	-	Body force
c_0	-	Speed of wave in the mixture
c_g	-	Group speed of the wave
c_l	-	Speed of sound in the liquid
c_p	-	Phase speed of the wave
C_1, C_2	-	Singular point at infinity
e_r	-	Radial component of the unit vector
G	-	Modulus of rigidity
$H(x, y)$	-	First integral
$\mathbf{I}$	-	Identity matrix
K	-	Cluster radius
$\mathbf{k}$	-	Wave number vector
k	-	Wave number of a transformed coordinate
k^x, k^y, k^z	-	Wave numbers in x, y and z directions
l	-	Characteristic length pf the wave
l_0	-	Width of the shock
$m_i,\ n_i,\ s_i\, r_i$	-	Coefficient of nonlinear equations $(i = 1, 2, \cdots)$
m	-	Effective mass of bubbly system
M	-	Maximum term in tanh-coth equation
n	-	Polytropic exponent of gas
N	-	Number of bubbles
Nu	-	Nusselt number
$\mathbf{n}$	-	Normal unit vector

o	-	Small oh notation
O	-	Big oh notation
p_0	-	Equilibrium pressure of the mixture
$p(x, y, z, t)$	-	Mixture pressure
$p_g(x, y, z, t)$	-	Pressure of gas
p_{g0}	-	Equilibrium pressure of gas
p_B	-	Pressure at the bubble boundary
p_d	-	Pressure at the finite domain
p_∞	-	Pressure far from the bubble
q	-	Heat flux
r	-	Distance from the bubble centre to a far point
R_0	-	Equilibrium bubble radius
R_d	-	Radius of finite domain
$R(t)$	-	Instantaneous bubble radius
$R(x, y, z, t)$	-	Bubble radius in space and time
t	-	Time
t_*	-	Characteristic collapse time
$\mathbf{T}$	-	Cauchy stress tensor
$\mathbf{x} = (x, y, z)$	-	Space coordinates
$v^{(1)} = v^{(1)}(x, y, z, t)$	-	Cartesian mixture velocity in x direction
$v^{(2)} = v^{(2)}(x, y, z, t)$	-	Cartesian mixture velocity in y direction
$v^{(3)} = v^{(3)}(x, y, z, t)$	-	Cartesian mixture velocity in z direction
V_0	-	Equilibrium volume of the bubble
V	-	Volume of the bubble
$\mathbf{V}(\mathbf{x}, t) = (v^{(1)}, v^{(2)}, v^{(3)})$	-	Mixture velocity in Cartesian coordinate
$\mathbf{V} = (v^r, 0, 0)$	-	Spherical velocity of the mixture
v^r	-	Radial velocity of the mixture
w	-	Wave frequency
w_0	-	Natural frequency of bubble
X_1, X_2	-	Local singular point

Calligraphy Font

$\mathcal{A}$	-	Simplification
$\mathcal{B}$	-	Simplification
$\mathcal{C}$	-	Simplification
$\mathcal{M}$	-	Bubble mass
$\mathcal{P}, \mathcal{Q}, \mathcal{R}$	-	Coefficients of Ricatti equation
$\mathcal{S}$	-	Surface area of the mixture
$\mathcal{S}_{\mathcal{B}}$	-	Surface of a bubble
$\mathcal{S}_{\infty}$	-	Surface of large and concentric spherical surface
$\mathcal{V}$	-	Volume of the of the mixture

Greek Symbols

α	-	Gas volume fraction
β_i	-	Material constants
γ_1, γ_2	-	Material constants
λ_1	-	Relaxation parameter
λ_2	-	Retardation parameter
Λ	-	Bubble-bubble interaction parameter
$\dot{\gamma}$	-	Rate of strain tensor
$\dot{\gamma}_{rr}$	-	Rate of strain in radial direction
Π	-	Divergent spherical wave
ν	-	Kinematic viscosity
ψ	-	Velocity potential
Ψ	-	Convergent spherical wave
η	-	Measure of dispersion
$\sum$	-	Summation
r, θ, ϕ	-	Spherical coordinates
μ	-	Dynamic viscosity of the liquid
ν	-	Thermal diffusivity coefficient of gas
Ω	-	Wave variable

κ	-	Rescaling factor for y
$\varkappa$	-	Rescaling factor for z
ξ, δ, ζ	-	Stretch directions (coordinates)
$\rho(x, y, z, t)$	-	Mixture density
ρ_0	-	Undisturbed density of the mixture
ρ_l	-	Density of liquid
ρ_g	-	Density of gas
σ	-	Surface tension
∇	-	$(\partial/\partial x, \partial/\partial y, \partial/\partial z)$ is a ”del” operator
Δ	-	$\nabla^2 = (\partial^2/\partial x^2, \partial^2/\partial y^2, \partial^2/\partial z^2)$
τ	-	Stretch time
$\tau_{\theta\theta}$	-	Polar shear stress at the bubble radius
τ_{rr}	-	Radial shear stress at the bubble radius
$\tau_{\phi\phi}$	-	Azimuthal shear stress at the bubble radius
$\boldsymbol{\tau}$	-	Extra stress tensor
$\varphi(x, y, z, t)$	-	Perturbation of the bubble radius
Φ	-	New wave coordinate
χ_g	-	Thermal conductivity coefficient of gas
ω	-	Isentropic exponent of gas
$\sim$	-	Asymptotic to

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CHAPTER 1

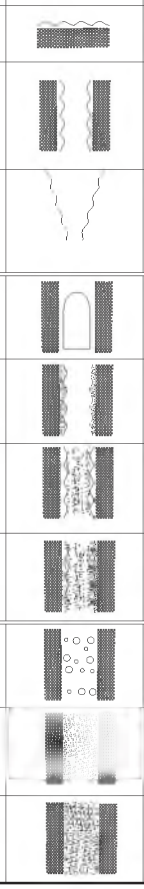
INTRODUCTION

This thesis is concerned with modified Rayleigh-Plesset equation (bubble dynamics equation) and the perturbation method to investigate the shock wave propagation in a bubbly pseudo-compressible viscoelastic liquid flow, incorporating the effect of bubble to bubble interaction. In this study, it is assumed that the population of bubbles is initially present in the viscoelastic liquid. These bubbles are normally spherical in shape and may interact with other. Therefore, the analysis of wave propagation in pseudo-compressible viscoelastic liquid with bubbles, together with inter bubble interaction, present the main aspect of this research.

In this chapter, the general overview of this thesis regarding the Rayleigh-Plesset equation and shock wave propagation in bubbly pseudo-compressible viscoelastic liquid flow will be discussed. Background of the research is presented in Section 1.1, while the problem statement is given in Section 1.2. The objectives of the research and scope of the study are highlighted in Section 1.3 and Section 1.4 respectively. The significance of the study is presented in Section 1.5, whereas the research methodology is given in Section 1.6. Finally, the thesis outline is given in Section 1.7.

1.1 Research Background

Many systems in both physical and biological sciences do not involve only single phase, such as gas, liquid or solid, but exist as a mixture of two or more



Class	Typical regimes	Geometry	Configuration	Examples
Separated flows	Film flow		Liquid film in gas Gas film in liquid	Film condensation Film boiling
	Annular flow		Liquid core and gas film Gas core and liquid film	Film boiling Boilers
	Jet flow		Liquid jet in gas Gas jet in liquid	Atomization Jet condenser
Mixed or Transitional flows	Cap, Slug or Churm-turbulent flow		Gas pocket in liquid	Sodium boiling in forced convection
	Bubbly annular flow		Gas bubbles in liquid film with gas core	Evaporators with wall nucleation
	Droplet annular flow		Gas core with droplets and liquid film	Steam generator
	Bubbly droplet annular flow		Gas core with droplets and liquid film with gas bubbles	Boiling nuclear reactor channel
Dispersed flows	Bubbly flow		Gas bubbles in liquid	Chemical reactors
	Droplet flow		Liquid droplets in gas	Spray cooling
	Particulate flow		Solid particulate gas or liquid	Transportation of powder

Figure 1.1 Two-phase flow inter-facial classification (Ishii and Hibiki, 2010)

phases. The components of a mixture could be gas-liquid, liquid-solid, liquid-liquid or gas-solid. Two-phase flow is very significant in phase-component-materials, which takes into account the flow configuration, behaviour as well as the interactions of phases. The occurrence of two-phase flow system indicates how significant a broad approach is needed in order to understand the flow behaviour. The complexity and difficulty of multi-phase flows have attracted a considerable amount of research in order to improve the understanding and knowledge regarding two-phase flow. The classification of two-phase flow depends on the interfacial structure of the two phases, such as dispersed flow, separated flow and transitional flow, which contributed to the difficulty in analysing two-phase flow systems. This classification is depicted in Figure 1.1 (Ishii and Hibiki, 2010).

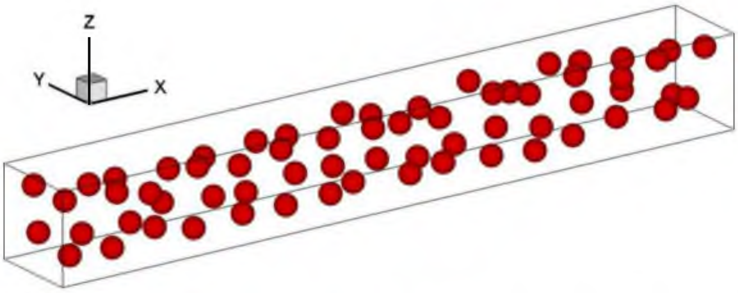


Figure 1.2 Bubbly-liquid flow diagram (Seo et al., 2010)

Bubbly-liquid flow is one of the types of two-phase flow. Despite the significance of two-phase flows, the understanding of bubbly-liquid flow is quite inadequate compared to single-phase flow. Therefore, the need to further understand the bubbly-liquid flow characteristics and behaviour represents the main interest of this research. Bubbly-liquid flow in three-dimensional case is shown in Figure 1.2.

The study of wave propagation in bubbly viscoelastic liquid is an interesting two-phase flow problem. Due to the high compressibility nature of gas in the bubbles, it can exhibit strongly nonlinear behaviour. The existence of small number of gas bubbles can significantly affect the acoustic properties of the liquid.

Bubbly-liquid occur in several physical, biological, industrial and engineering fields. For example, Leighton (2015) explained that due to the air becoming entrained in the ocean by activities of the passing ships, bubbles are form near the ocean surface area in the form of bubble clusters. The bubbles impact on the aquatic life and marine environment, in the middle and surface of the ocean is tremendous. Ichihara et al. (2004) analyses the seismic wave-producing bubbles in magma, where bubbles tend to be accumulated in the upper layers and topmost part of magma reservoirs, given rise to a pressure that can easily cause an abruptly discharging volcanic eruption. Kedrinskiy (2006) discovered that bubbles radial and volumetric oscillations result in the underwater bubbles explosion. Sonoluminescence (the emission of short bursts of light from imploding bubbles in a liquid when excited by sound) is also a broad area that has wide applicability of bubbly flows (Puente and Bonetto, 2005; Luo et al., 2015). Similarly, Storey and Szeri (2000) considered a case where acoustically forced bubbles create high temperatures and pressures during collapse, resulting in what is termed Sonochemistry, that enhances the chemical reaction and combination of substances within or near the bubbles .

Cunliffe et al. (2013) states that the existence of bubbles close to the surface of an ocean poses a lot of benefits that are both biological and environmental, which include the release of gas from the ocean and atmosphere. Shklyaev and Straube (2010) and Dugue et al. (2013) indicate that bubble screens are used to damp shocks produced by underwater bubble cloud explosions. That is because bubbly-liquid provides protection from shocks due to the dissipative nature of the bubbles.

Scardina and Edwards (2001) assert that bubbly-liquid occurs commonly in industry, whose presence can lead to a deficiency in terms of efficiency, for example, in boilers, chemical reactors and hydraulic devices. Scardina and Edwards (2001) ascertain that a poor understanding of the bubble-liquid mixture mechanics of a system may bring about safety issues that arise in industries that use very complex piping configurations, such as nuclear power plants. The benefit of bubbly-liquid flow can also be seen in medical ultrasound, such as lithotripsy (Zhong, 2013) and as micro-bubble contrast agents in imaging (Yang et al., 2018). Bubbles in constrain spaces do work as an efficient pump. This gives them the potential to be used as a drug delivery mechanism (Kooiman et al., 2014).

The important study of nonlinear wave propagation in bubbly-liquid mixture was carried out by Wijngaarden (1968), where he described the mixture as a continuum, Wijngaarden's derivation of the model was based on intuitive physical arguments and derived Boussinesq and Korteweg-de-Vries (KdV) type equations, where the incompressible liquid is assumed to have no any damping effects (viscosity, thermal, acoustic) and also no surface tension effect. Wijngaarden (1972) included the effect of viscosity in a bubbly incompressible liquid to derived weakly nonlinear waves evolution equation; the Korteweg-de-Vries and Burgers (KdVB) equation in a one-dimensional case. The effect of variations in bubbles velocities on the waves propagating in bubbly liquids was studied in Mond (1987), It was shown that waves are weakly damped even in a non-dissipative way (in the absence of viscosity). The thermal exchange between the bubble and liquid and the condition that provides the dominant damping mechanism during bubble oscillations was considered in Watanabe and Prosperetti (1994).

Drumheller and Bedford (1979) developed a model in which the liquid and the bubbles are treated as separate continua (the bubble and the liquid are moving with different velocities) with viscosity and surface tension. Fusco and Oliveri (1989) used a variational procedure to obtain the equations of motion to check the validity of the model. Oliveri (1989) proposed model equations to a non-diffusive bubbly-liquid and used asymptotic analysis to study the nonlinear wave propagation, where the gas volume fraction is small. Wave Modulation equations for planar one-dimensional bubbly medium with an incompressible carrier phase without surface tension and no damping mechanism was derived by Gavrilyuk (1989), and taken into account small viscous term and interphase heat exchange in Gumerov (1994).

It is worth mentioning that the nonlinear evolution equations mentioned above were derived with the condition that the surrounding liquid is purely viscous and incompressible. Nonlinear wave propagation in pseudo-compressible bubbly viscoelastic liquid (polymer solution, magma, syrup, suspensions) flows accounting for inter bubble interaction has not been investigated, which gave the motivation to consider the effect of viscoelasticity in monodisperse bubbly-liquid flow.

1.2 Problem Statement

Bubbly-liquid flow involves some relative motions of the bubble phase with respect to the liquid. This bubbly-liquid when flowing are influenced by the rheological properties of the bubble and liquid phases, and the interaction of the bubbles as well as the volume fraction of each of gas bubble and liquid phases. The existence of any amount of gas bubble makes the system a two-phase flow. The effects of compressibility of the bubble on the velocity of the wave propagation in bubbly flow and on the pressure changes are important to be considered in the analysis of bubbly-liquid flow. This is due to the fact that even the smallest amount of gas bubbles in flow affect the wave propagation.

This bubbly-liquid flow is a complex two-phase flow due to the compressibility nature of the gas phase. Due to the complex nature of the flow, the solution (both numerical and analytical) techniques of the bubbly flow is a very difficult task. In bubbly-liquid flow, viscoelastic liquids are mostly encountered. The viscoelasticity occurs naturally in the liquid properties or as an additive to the liquid which affects or causes changes in pressure or temperature at any location in the liquid flow. Bubbly-liquid flow is treated as either having inviscid or viscous liquid characteristics. Bubbly-liquid flow in real-life application are by their nature not only inviscid or viscous liquid, and any point in the bubbly-liquid flow is seen to have alternating high and low gas bubble fractions due to the viscoelastic properties of the liquid. These facts indicate that an improved bubbly-liquid flow model should be a viscoelastic model, that is, it should have both the Newtonian and non-Newtonian properties of the liquid under consideration. As bubbly-liquid flows, heat is constantly transferred to bubble from the surrounding temperature of the liquid and as a result, the temperatures of the bubbly-liquid changes. Therefore, this temperature fluctuation affects the flow behaviour. Some researchers on bubbly-liquid flow assumed that temperature is constant in the flow, thereby neglecting the energy properties of the system.

Many analytical and semi-analytical solution techniques have been developed recently to solve partial differential equations arising from mathematical models of bubbly-liquid flow. In most models of bubbly-liquid flow, the construction of solution is complex because of the nature of nonlinear, dispersive and dissipative terms. These lead to numerous difficulties that result to the use of stretching coordinate transformation with a lot of limitations. Hence, in improving computational accuracy for the numerical solution in bubbly-liquid flow, Adomian decomposition method has been found capable to predict accurately the flow characteristics in transient bubbly-liquid flow. This method has mostly been applied in the other areas of sciences. This method is yet to be used in a bubbly-liquid flow problem.

The problem statement can be summarised as follows:

- Bubble dynamics equation needs to be modified to account for the effect of liquid pseudo-compressibility, viscoelasticity and bubble to bubble interaction. Hence the modification of Rayleigh–Plesset(RP) equation.
- pseudo-compressibility, bubbly viscoelasticity and bubble-bubble interaction are encountered in liquid flow, hence the need to investigate their effect on shock wave propagation of the mixture.
- Some analytical method for solving shock wave models give limited solutions, hence the need for improvement.

1.3 Research Objectives

Based on the problems stated above, this study focuses to propose a theoretical model that considers the bubbles interaction, pseudo-compressible viscoelastic liquid with bubbles with the following specific objectives :

1. To propose a modified Rayleigh–Plesset equation that describes the bubble dynamics in a pseudo-compressible, viscoelastic liquid together with bubble-bubble interaction.
2. To adopt the perturbation method in investigating the effects of pseudo-compressible bubbly viscoelastic liquid flow with bubble-bubble interaction on the shock wave propagation.

3. To check the existence of shock wave solutions and improve some analytical methods for solving shock wave model equations.

1.4 Scope of the Study

A modified Rayleigh–Plesset equation for bubbly pseudo-compressible viscoelastic (both linear and nonlinear) liquid flow that accounts for inter bubble interactions, with polytropic and heat transfer is considered in this research. A homogeneous model is used for the bubbly viscoelastic liquid flow. Two different viscoelastic liquids, linear (Kelvin-Voigt) and nonlinear (second-grade) are considered for pressure, velocity and bubble radius perturbations of the bubbly-liquid flow. The flow models are formulated with the assumption that bubbles and liquid are coupled together as a continuum and there is interphase interaction between liquid and bubbles. There is no mass transfer between the liquid and bubble, the buoyancy effect is neglected. The model equations are reduced to a single PDE using the perturbation method. The existence of shock wave solution, local and global stabilities for the single PDE are analysed. Methods of obtaining both the real and complex travelling wave solutions are improved using modified tanh-coth method combined with Riccati equation, and also problems with forcing terms.

1.5 Significance of Findings

In this thesis, the shock wave propagation in bubbly pseudo-compressible viscoelastic liquids flow, accounting for bubble-bubble interaction in both the polytropic and heat transfer cases are studied. The model equations are reduced to a nonlinear evolution equation whose solutions are derived and analysed. The significance of this research is due to the fact that incompressible viscoelastic liquid flow does not quite describe the wave propagation in bubbly viscoelastic liquid flow and therefore has not been quite successful in describing the wave propagation in bubbly-liquid flow. This difficulty has now been resolved by a pseudo-compressible viscoelastic liquid assumption. The rheological concept of compressible liquid is of special importance due to its application in many engineering and industrial applications.

This study will be beneficial to many industrial applications. In view of the flow transportation system, an accurate understanding of volume fractions, bubbly-liquid pressure fluctuation and heat transfer coefficient which are considered as significant parameters for economical design and operation will be explained. This research will give an important understanding of the flow phenomenon in the bubbly viscoelastic liquid systems, since the bubble nonlinear dynamics is known to greatly influence the bubbly viscoelastic liquid flow, that leads to a change in physical characteristics of the flow patterns, and eventually alters the magnitudes of the velocity and pressure wave propagation.

Generally, since it's now possible for bubbles to interact without breakage or formation, which more often occur in bubbly flow scenario, this research will give a vital understanding of the flow phenomenon in the bubbly system, that is known to substantially influence the bubble-liquid flow, which leads to a change in physical properties of the flow patterns.

1.6 Research Methodology

The research methodology adopted to achieve the outlined objectives of this study of modified Rayleigh–Plesset equation, and shock wave propagation in bubbly pseudo-compressible viscoelastic liquid using perturbation method, is given in broad terms in Fig 1.3, while the overall research design is depicted in Fig 1.4.

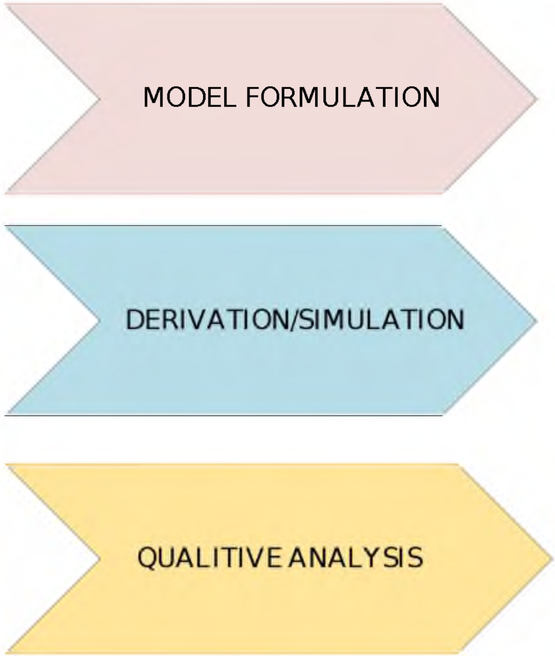


Figure 1.3 The research methodology in general

1.6.1 Pseudo-Compressibility

The mathematical characteristic of governing flow equation used for incompressible liquids is changed from elliptic dominated to hyperbolic dominated, by applying artificial compressibility concept. Resorting to the pseudo-compressibility concept, the continuity constraint is perturbed by the time derivative of density. This approach introduces a variable in the Rayleigh–Plesset equation for bubble dynamics—pseudo-pressure, allowing one to use the slightly or weakly compressible liquid.

1.6.2 Mathematical Formulation

The mathematical formulation, using the mass and momentum conservation equations of the governing equation for the problem outlined in the objectives of this study is derived. They are coupled with the equation for the mixture density, that is the sum of the liquid and gas bubble densities. A modified general bubble dynamics (Rayleigh–Plesset) equation for compressible viscoelastic liquid with bubble interaction is derived using the energy conservation equation (Prosperetti, 1987; Doinikov, 2005). Equation of state for gases is coupled to the equation for the bubble dynamics. Two types of viscoelastic liquids will be considered in the governing equation, which are the Kelvin-Voigt and second grade liquids. A three-dimension flow is considered in the model formulation. The model equations are linearised to obtain the non-dispersive wave equations and the speed of wave in the bubbly-liquid. Using non-dimensional variables, each equation is written in a non-dimensional form. Both the dispersive equation and dispersion relation are obtained in terms of the linearised non-dimensional governing equations.

1.6.3 Derivation and Solutions of Nonlinear Wave Equations

Reduction perturbation method (techniques) is adopted to the general nonlinear version of the non-dimensional governing equations, using the stretched coordinates of the space and time variables. Asymptotic expansions on the field quantities (pressure, velocity and radius perturbation of the bubble) will be used to reduce the system to a single three dimensional nonlinear wave equation ((3+1)-KdVB equation). This is done by equating the corresponding powers of the perturbation parameter (ϵ) and solving the resulting equation. Adomian decomposition will be used to derived semi-analytical solutions to the equation. Qualitative analysis of the KdVB equation is carried out, where both the local and global stability analysis are conducted. The local and global stability at both finite and infinite equilibrium points are investigated using the Poincare transformation and the global phase portrait. A modified analytical solutions (real and complex) are derived for the (3+1)-KdV and (3+1)-KdVB equations.

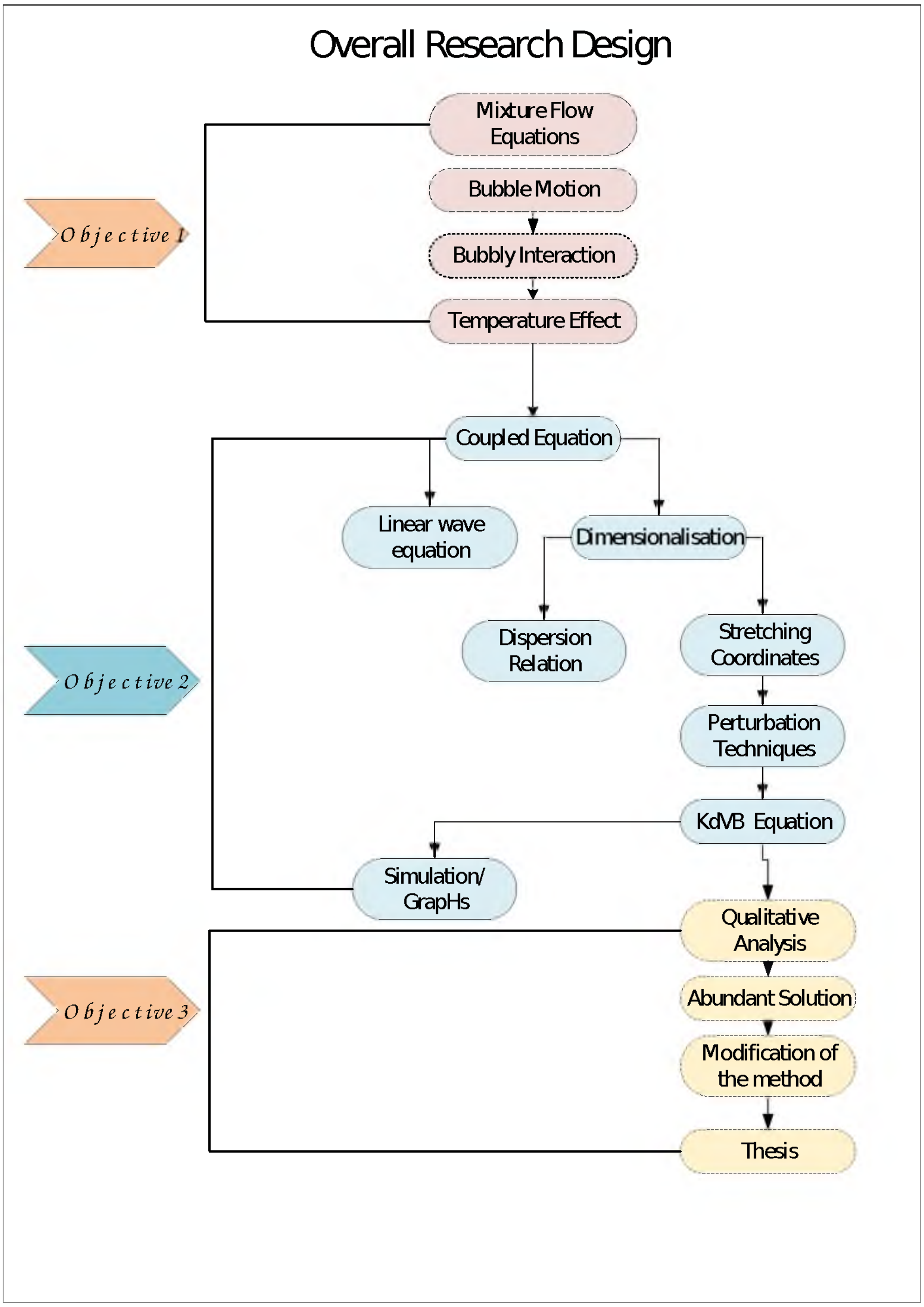


Figure 1.4 Overall research design

1.6.4 Asymptotic Expansion

This branch of mathematics has a reasonably long history (Johnson, 2006). First we require a little bit of notations

$$f(x) = o(g(x)), \quad f(x) = O(g(x)), \quad f(x) \sim g(x) \tag{1.1}$$

as $x \to x_0$, if

$$\lim_{x \to \infty} \frac{f(x)}{g(x)}$$

is zero, finite non-zero or unity respectively (Johnson, 2006). These are usually read as 'little oh', 'big oh' and 'varies as' (or 'asymptotically equal to'), respectively; the function $f(x)$ is the given function under discussion, and $g(x)$ is a suitable gauge function. This description of a function (in a limit) is now extended: we write

$$f(x) - \sum_{n=0}^{N-1} g(x) \sim g_N(x) \quad \text{as} \quad x \to x_0, \tag{1.2}$$

for every $N \geq 1$, where $f(x) \sim g_0(x)$ as $x \to x_0$. It is then usual (and convenient) to express this property in the form

$$f(x) - \sum_{n=0}^{\infty} g_n(x) \quad \text{as} \quad x \to x_0, \tag{1.3}$$

where N has been taken to infinity here; this 'series' is called an asymptotic expansion of $f(x)$, as $x \to x_0$. Asymptotic expansions are rarely taken beyond a few terms, but it is be possible to find them all.

1.6.5 Perturbation Method

This part introduces the basic perturbation method. The theory involves mathematical methods for finding series expansion approximations for perturbed systems. Perturbation theory can be applied to algebraic equations, boundary value problems, difference equations, Hamiltonian systems, ODEs and PDEs. The main idea begins with the assumption that the solution to the perturbed can be expressed as an asymptotic expansion, that is perturbation method is a method of constructing an asymptotic expansion.

The most straightforward type of non-uniformity, arises when the asymptotic expansion that has been obtained breaks down and thereby leads to the introduction of a new scaled variables. This situation is typical of some wave propagation problems, for which an asymptotic expansion valid near the initial data becomes non-uniform for later times/large distances (Johnson, 2006). From the solution of linear non-dispersive wave, the right-going wave will be followed (by selecting any $\mathbf{k} \cdot \mathbf{x} - w\,t =$ constant) then, as t increases indefinitely, we will encounter a breakdown when $\epsilon\,t = O(1)$. This will lead to introduction of new variables (scaled variables) $(\xi, \delta, \zeta, \tau)$. Thus, we transform from (x, y, z, t) variables (the near-field) to ξ, δ, ζ, τ variables (the far-field).

The method of transformation from near-field to far-field variables of a higher dimensional system is introduced and applied in Washimi and Taniuti (1966), Kako and Rowlands (1976) and Kudryashov and Sinelshchikov (2012). Using change of coordinates via chain rule, various derivatives will be obtained which is then substituted in to the continuity, momentum equations in component forms and also into the simplified modified Rayleigh–Plesset equation. The asymptotic expansion of the field variables will then be derived using the scaled variables. Solution to the scaled model equations will be sought in the form of the asymptotic series expansions in terms of asymptotic parameter ϵ. The resulting equations are collected according to the degree of ϵ, and then simplify to obtain a nonlinear evolution equation.

1.6.6 Adomian Decomposition Method

Adomian (1991) developed the Adomian decomposition method which has receiving much attention in recent years in applied mathematics. The method proved to be powerful, effective, and can easily handle a wide class of ordinary and partial differential equations. The method demonstrates fast convergence of the solution which provides significant advantages. The method will be successfully used to handle the partial differential equations that will be derived in this research. The method solves a problem without using linearisation, perturbation or any other restrictive assumption that may change the behaviour of the model under consideration. More detail explanations of the method can be found in Adomian (1991), Wazwaz (2001), Wazwaz (2006) and the references therein.

Below is the description of the Adomian decomposition method to be adopted in this thesis. Consider the differential equation of the form

$$Lv + Rv + Nv = g, \tag{1.4}$$

where Lv is the highest order derivative which is assumed to be easily invertible, Rv is a linear differential operator of order less than Lv, Nv represents the nonlinear terms, and g is the source term. Applying the inverse operator L^{-1} to both sides of (1.4), and using the given conditions, then

$$v = f - L^{-1}(Rv) - L^{-1}(Nv), \tag{1.5}$$

where the function f represents the terms arising from integrating the source term g and from using the given conditions, all are assumed to be prescribed. The standard Adomian decomposition method defines solution by the series

$$v(x) = \sum_{n=0}^{\infty} v_n(x), \tag{1.6}$$

where the components $v_0, v_1, v_2, \cdots$ are usually determined recursively by

$$\begin{aligned} v_0 &= f, \\ v_{k+1} &= -L^{-1}(Rv_k) - L^{-1}(Nv_k), \qquad k \geq 0. \end{aligned} \tag{1.7}$$

The decomposition method suggests that the zeroth component v_0 usually defined by the function f described above. After determining the components $v_0, v_1, v_2, \cdots$, the solution v in a series form defined in (1.6) follows immediately.

The Adomian scheme for calculating nonlinear terms will be introduced here. However, the nonlinear term Nv, such as $v^2, v^3, v^4, \sin v, e^v, vv_x, v_x^2$, can be expressed by an infinite series of the so-called Adomian polynomials B_n given in the form

$$F(v) = \sum_{n=0}^{\infty} B_n(x)(v_0, v_1, v_2, \cdots, v_n), \tag{1.8}$$

where the so-called Adomian polynomials B_n can be evaluated for all forms of nonlinearity. Several schemes have been introduced in the literature by researchers to calculate Adomian polynomials.

The Adomian polynomials B_n for the nonlinear term $F(v)$ can be evaluated by using the following expression

$$B_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[F\left(\sum_{i=0}^{n} \lambda^i v_i\right)\right]_{\lambda=0} \qquad i = 0, 1, 2, \cdot \tag{1.9}$$

The general formula (1.9) can be simplified as follows. if the nonlinear function is $F(v)$, then by using (1.9), Adomian polynomials are given as (Wazwaz, 2010a)

$$\begin{aligned}
B_0 &= F(v_0),\\
B_1 &= v_1 F'(v_0),\\
B_2 &= v_2 F'(v_0) + \frac{1}{2!}v_1^2 F''(v_0),\\
B_3 &= v_3 F'(v_0) + v_1 v_2 F''(v_0) + \frac{1}{3!}v_1^3 F'''(v_0),\\
B_4 &= v_4 F'(v_0) + \left(\frac{1}{2!}v_2^2 + v_1 v_3\right) F''(v_0) + \frac{1}{2!}v_1^2 v_2 F'''(v_0) + \frac{1}{4!}v_1^4 F^{(4)}(v_0).
\end{aligned} \tag{1.10}$$

Other polynomials can be generated in a similar manner.

1.6.7 Graphical Results and Discussions

The graphical representation of results are very important because they not only explain the physical properties of the problem but also verifies the solutions. Having this motivation, the solution in each problem is displayed graphically. The properties and effects of the most important parameters on the shock wave propagation will be investigated. Maple software will be used to plot the graphs.

1.6.8 Tanh-Coth Method with Riccati Equation

Here, a brief description of tanh-coth method combined with Riccati equation to solve the would-be derived partial differential equation would be given. The would-be derived nonlinear evolution equation will be reduced to an ODE via a wave coordinate transformation $\Phi = k(\xi + \delta + \zeta - c\tau)$. The ODE can be solved using tanh-coth method combined with Riccati equation (Wazzan, 2009), which admits the use of finite expansion

$$v(Y) = \sum_{i=0}^{M} a_i Y^i + \sum_{i=1}^{M} b_i Y^{-i}, \tag{1.11}$$

with the Riccati equation

$$Y' = \mathcal{P} + \mathcal{Q}Y + \mathcal{R}Y^2. \tag{1.12}$$

By changing of variable

$$\begin{aligned} \frac{d}{d\Phi} &= \left(\mathcal{P} + \mathcal{Q}Y + \mathcal{R}Y^2\right)\frac{d}{dY}, \\ \frac{d^2}{d\Phi^2} &= \left(\mathcal{P} + \mathcal{Q}Y + \mathcal{R}Y^2\right)\left((\mathcal{Q} + 2\mathcal{R}Y)\frac{d}{dY} + \left(\mathcal{P} + \mathcal{Q}Y + \mathcal{R}Y^2\right)\frac{d^2}{dY^2}\right), \end{aligned} \tag{1.13}$$

where $\mathcal{P}$, $\mathcal{Q}$ and $\mathcal{R}$ are real numbers to be given, while a_i and b_i are constants to be determined later. The positive integer M can be determined by considering the homogeneous balance (Rady et al., 2010) between the highest order derivatives and the most nonlinear terms appearing in the ODE. If M is not an integer, then a transformation formula should be used to overcome the difficulty. Substituting (1.11) into the ODE and making use of (1.12) and (1.13) yield an equation in terms of Y^i. Equating the coefficients of all the powers of Y^i to zero, we obtain a set of algebraic equations for $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{R}$, a_i, b_i, c and k. From the aforementioned steps, expansion (1.13) reduces to the standard tanh-coth method. In this work, we shall use the following solutions of the Riccati equation (1.12) :

$\mathcal{P} = \mathcal{Q} = 1$ and $\mathcal{R} = 0$, then $Y' = 1 + Y$ and solving to obtain

$$Y = \exp(\Phi) - 1. \tag{1.14}$$

$\mathcal{P} = \frac{1}{2}$, $\mathcal{Q} = 0$, $\mathcal{R} = -\frac{1}{2}$, then $Y' = \frac{1}{2}(1 - Y^2)$, solving this

$$\begin{aligned} \frac{dY}{d\Phi} &= \frac{1}{2}\left(1 - Y^2\right), \\ \frac{dY}{1 - Y^2} &= \frac{1}{2} d\Phi, \\ \tanh^{-1} Y &= \frac{1}{2}\Phi, \\ Y &= \tanh\left(\frac{1}{2}\Phi\right) = \frac{\sinh\Phi}{\cosh\Phi + 1} = \frac{\cosh\Phi - 1}{\sinh\Phi} = \coth\Phi - \operatorname{csch}\Phi. \end{aligned} \tag{1.15}$$

Other solutions of (1.12), such as $Y = \coth\Phi + \operatorname{csch}\Phi$ and $Y = \tanh\Phi \pm i \operatorname{sech}\Phi$ can be found in Wazzan (2009).

1.6.9 Secant Hyperbolic Method with Riccati Equation

In this thesis, the secant hyperbolic ansatz (an assumption about the form of an unknown function which is made in order to facilitate solution of an equation or other problem) will be improved and used to derive many complex solutions to the (3+1)-KdV and (3+1)-KdVB equations. The first step is the transformation of the nonlinear PDE to a nonlinear ODE via coordinate transformation. Introducing the secant hyperbolic ansatz

$$v(\Phi) = g(\Phi) + h(\Phi) \operatorname{sech}(\Phi), \qquad \Phi = k(\xi + \delta + \zeta - c\,\tau) \tag{1.16}$$

where $g(\Phi)$ and $h(\Phi)$ are functions to be determined later. The derivatives of (1.16) are

$$\begin{aligned} v^{'}(\Phi) &= g^{'} + \Big(h^{'} - h\tanh(\Phi)\Big)\operatorname{sech}(\Phi), \\ v^{''}(\Phi) &= g^{''} + \Big(h^{''} + 2\,h\tanh^{2}(\Phi) - 2\,h^{'}\tanh(\Phi) - h\Big)\operatorname{sech}(\Phi). \end{aligned} \tag{1.17}$$

Therefore, equations (1.16) and (1.17) will be substituted into the nonlinear ODE, the resulting ODE will be solve using tanh-coth method with Riccati equation discussed in the above section.

1.7 Thesis Organization

This thesis contains seven chapters. It is mainly concerned with the modified Rayleigh-Plesset equation and shock wave propagation in bubbly pseudo-compressible viscoelastic liquid flow, incorporating the effect of bubble to bubble interaction with polytropic and heat transfer. The first chapter serves as an introduction to the whole thesis. Chapter 1 introduces the background of the study, which gives a general introduction, followed by the statement of the problem, objectives and scope of the research. Significance of the study and methodology adopted in achieving the outlined objectives of the present research are all covered in the first chapter.

In Chapter 2, a literature review of this research regarding to the problem outlined in the objectives of this research are given. More precisely, the literature on bubble dynamics in viscous and viscoelastic liquids, thermodynamics of gas bubble, bubble-bubble interaction, shock wave propagation in bubbly-liquid flow dynamics etcetera. Various works by different researchers regarding these topics are cited.

Chapter 3 concerns with the derivation of governing equations describing bubbly pseudo-compressible viscoelastic liquid flow. This includes the derivation of a modified Rayleigh–Plesset equation in compressible viscoelastic liquid using the method of Kinetic energy, and with bubble-bubble interaction with two different heats scenario.

Chapter 4 discusses the transient bubbly Kelvin-Voigt liquid flow in an unbounded space, where the bubble dynamics in Kelvin-Voigt liquid is derived and incorporated into the equation of bubbly-liquid mixture, the equations are non-dimensionalised. A dispersive and non-dispersive linear wave equations are obtained. A perturbation method and asymptotic expansion are used to derive a nonlinear wave equation. Adomian decomposition method is used to obtained a semi-analytical solution. Graphical solutions and discussion of results are also given.

Chapter 5 contains the transient bubbly second-grade liquid flow in an unbounded space, where the bubble dynamics in a second-grade liquid is derived and incorporated into the equations of bubbly-liquid mixture. A dispersive and non-dispersive linear wave equations are obtained. Asymptotic expansion is used to derive a nonlinear wave equation. Semi-analytical solution using Adomian decomposition method are derived. Graphical representation and discussion of the solutions are given.

Chapter 6 deals with the existence of solution via qualitative analysis of the local and global equilibrium points of the derived (3+1)-KdV and (3+1)-KdVB equations. Also, abundant real and complex exact solutions to the derived nonlinear evolution equations are obtained, using the modified tanh-coth and modified secant hyperbolic methods, combined with Riccati equations.

In Chapter 7, conclusions and recommendations for future research are highlighted.

REFERENCES

Adomian, G. (1991). A review of the decomposition method and some recent results for nonlinear equations, *Computers & Mathematics with Applications* **21**(5): 101–127.

Al-Sarkhi, A. (2010). Drag reduction with polymers in gas-liquid/liquid-liquid flows in pipes: a literature review, *Journal of Natural Gas Science and Engineering* **2**(1): 41–48.

Albernaz, D. and Cunha, F. (2011). Bubble dynamics in a maxwell fluid with extensional viscosity, *Mechanics Research Communications* **38**(3): 255–260.

Allen, J. S. and Roy, R. A. (2000a). Dynamics of gas bubbles in viscoelastic fluids. i. linear viscoelasticity, *The Journal of the Acoustical Society of America* **107**(6): 3167–3178.

Allen, J. S. and Roy, R. A. (2000b). Dynamics of gas bubbles in viscoelastic fluids. ii. nonlinear viscoelasticity, *The Journal of the Acoustical Society of America* **108**(4): 1640–1650.

Amore, P. and Fernández, F. M. (2013). Mathematical analysis of recent analytical approximations to the collapse of an empty spherical bubble, *The Journal of chemical physics* **138**(8): 084511.

Ando, K., Colonius, T. and Brennen, C. E. (2011). Numerical simulation of shock propagation in a polydisperse bubbly liquid, *International Journal of Multiphase Flow* **37**(6): 596–608.

Ando, K., Colonius, T. and Brennen, C. E. (2013). Shock propagation in polydisperse bubbly liquids, *Bubble Dynamics and Shock Waves*, Springer, pp. 141–175.

Askari, V., Sadeghy, K. and Tavanashad, V. (2013). Translational motion of non-spherical cavitation bubbles collapsing in a viscoelastic fluid near a rigid boundary, *Nihon Reoroji Gakkaishi* **41**(2): 53–65.

Barbat, T., Ashgriz, N. and Liu, C.-S. (1999). Dynamics of two interacting bubbles in an acoustic field, *Journal of Fluid Mechanics* **389**: 137–168.

Bjerknes, C. A. (1915). *Hydrodynamische Fernkräfte: fünf Abhandlungen über die Bewegung kugelförmiger Körper in einer inkompressiblen Flüssigkeit (1863-1880)*, number 195, W. Engelmann.

Bjerknes, V. F. K. (1906). *Fields of Force.*, Columbia University Press.

Bondarenko, A. G. and Kudryashov, N. A. (1989). Nonlinear waves in a liquid containing gas bubbles with allowance for phase transition, *Fluid Dynamics* **24**(3): 421–426.

Boubir, B., Triki, H. and Wazwaz, A. (2013). Bright solitons of the variants of the novikov–veselov equation with constant and variable coefficients, *Applied Mathematical Modelling* **37**(1-2): 420–431.

Brennen, C. E. (2005). *Fundamentals of multiphase flow*, Cambridge university press, London.

Brennen, C. E. (2013). *Cavitation and bubble dynamics*, Cambridge University Press, London.

Brujan, E. (2010). *Cavitation in non-Newtonian fluids with biomedical and bioengineering applications*, Springer Science & Business Media, Berlin.

Brujan, E. A. (2001). The equation of bubble dynamics in a compressible linear viscoelastic liquid, *Fluid Dynamics Research* **29**(5): 287.

Caflisch, R. E., Miksis, M. J., Papanicolaou, G. C. and Ting, L. (1985a). Effective equations for wave propagation in bubbly liquids, *Journal of Fluid Mechanics* **153**: 259–273.

Caflisch, R. E., Miksis, M. J., Papanicolaou, G. C. and Ting, L. (1985b). Wave propagation in bubbly liquids at finite volume fraction, *Journal of Fluid Mechanics* **160**: 1–14.

Carstensen, E. L. and Foldy, L. L. (1947). Propagation of sound through a liquid containing bubbles, *The Journal of the Acoustical Society of America* **19**(3): 481–501.

Cartmell, J. J. (2001). A continuum description of wave propagation in bubbly liquids with relative motion.

Cheng, L. and Zhang, Y. (2017). A kdv-type wronskian formulation to generalized kp, bkp and jimbo–miwa equations, *Communications in Theoretical Physics* **68**(1): 1.

Commander, K. W. and Prosperetti, A. (1989). Linear pressure waves in bubbly liquids: comparison between theory and experiments, *The Journal of the Acoustical Society of America* **85**(2): 732–746.

Cunliffe, M., Engel, A., Frka, S., Gašparović, B., Guitart, C., Murrell, J. C., Salter, M., Stolle, C., Upstill-Goddard, R. and Wurl, O. (2013). Sea surface microlayers: A unified physicochemical and biological perspective of the air–ocean interface, *Progress in Oceanography* **109**: 104–116.

Dealy, J. M. and Wissbrun, K. F. (2012). *Melt rheology and its role in plastics processing: theory and applications*, Springer Science & Business Media.

Delale, C. F., Nas, S. and Tryggvason, G. (2005). Direct numerical simulations of shock propagation in bubbly liquids, *Physics of Fluids* **17**(12).

Delale, C. F., Schnerr, G. H. and Sauer, J. (2001). Quasi-one-dimensional steady-state cavitating nozzle flows, *Journal of Fluid Mechanics* **427**: 167–204.

Delale, C. F. and Tryggvason, G. (2008). Shock structure in bubbly liquids: comparison of direct numerical simulations and model equations, *Shock waves* **17**(6): 433–440.

Demiray, H. (2005). A complex travelling wave solution to the kdv–burgers equation, *Physics Letters A* **344**(6): 418–422.

Doinikov, A. A. (2005). Equations of coupled radial and translational motions of a bubble in a weakly compressible liquid, *Physics of Fluids* **17**(12): 128101.

Doinikov, A. A., Novell, A., Escoffre, J.-M. and Bouakaz, A. (2013). Encapsulated bubble dynamics in imaging and therapy, *Bubble Dynamics and Shock Waves*, Springer, pp. 259–289.

Doinikov, A. and Zavtrak, S. (1995). On the mutual interaction of two gas bubbles in a sound field, *Physics of Fluids* **7**(8): 1923–1930.

Drumheller, D. S. and Bedford, A. (1979). A theory of bubbly liquids, *The Journal of the Acoustical Society of America* **66**(1): 197–208.

Dugue, V., Blanckaert, K., Qiuwen, C. and Schleiss, A. J. (2013). Reduction of bend scour with an air-bubble screen–morphology and flow patterns, *International Journal of Sediment Research* **28**(1): 15–23.

Dumortier, F., Llibre, J. and Artés, J. C. (2006). *Qualitative theory of planar differential systems*, Springer.

Elishakoff, I. (2001). Dynamical systems with applications using maple.

Fan, E., Zhang, J. and Hon, B. Y. C. (2001). A new complex line soliton for the two-dimensional kdv–burgers equation, *Physics Letters A* **291**(6): 376–380.

Feng, Y.-L., Shan, W.-R., Sun, W.-R., Zhong, H. and Tian, B. (2014). Bifurcation analysis and solutions of a three-dimensional kudryashov–sinelshchikov equation in the bubbly liquid, *Communications in Nonlinear Science and Numerical Simulation* **19**(4): 880–886.

Feng, Z.-s. and Meng, Q.-g. (2007). Burgers-korteweg-de vries equation and its traveling solitary waves, *Science in China Series A: Mathematics* **50**(3): 412–422.

Ferrara, K., Pollard, R. and Borden, M. (2007). Ultrasound microbubble contrast agents: fundamentals and application to gene and drug delivery, *Annu. Rev. Biomed. Eng.* **9**: 415–447.

Filiz, A., Ekici, M. and Sonmezoglu, A. (2014). -expansion method and new exact solutions of the schrödinger-kdv equation, *The Scientific World Journal* **2014**.

Fogler, H. and Goddard, J. (1970). Collapse of spherical cavities in viscoelastic fluids, *Physics of Fluids* **13**(5): 1135.

Fox, F. E., Curley, S. R. and Larson, G. S. (1955). Phase velocity and absorption measurements in water containing air bubbles, *The Journal of the Acoustical Society of America* **27**(3): 534–539.

Fusco, D. and Oliveri, F. (1989). Derivation of a non-linear model equation for wave propagation in bubbly liquids, *Meccanica* **24**(1): 15–25.

Fuster, D. (2018). A review of models for bubble clusters in cavitating flows, *Flow, Turbulence and Combustion* pp. 1–40.

Fuster, D. and Colonius, T. (2011). Modelling bubble clusters in compressible liquids, *Journal of Fluid Mechanics* **688**: 352–389.

Fuster, D., Hauke, G. and Dopazo, C. (2010). Influence of the accommodation coefficient on nonlinear bubble oscillations, *Journal of the Acoustical Society* **128**: 5–10.

Gasenko, V. and Nakoryakov, V. (2008). Nonlinear three-wave equation for a polydisperse gas-liquid mixture, *Journal of Engineering Thermophysics* **17**(3): 158–165.

Gaudron, R., Warnez, M. and Johnsen, E. (2015). Bubble dynamics in a viscoelastic medium with nonlinear elasticity, *Journal of Fluid Mechanics* **766**: 54–75.

Gavrilyuk, S. (1989). Modulation equations for a bubbly medium with an incompressible carrier phase, *Zh. Prikl. Mekh. Tekh. Fiz.* (2): 86–92.

Guan, K. and Gao, G. (1987). The qualitative theory of the mixed korteweg–de vries–burgers equation, *Sci. Sinica (Ser. A)* **30**: 64–73.

Gubaidullin, A., Beregova, O. S. and Bekishev, S. (2001). Shock waves in non-newtonian bubbly liquids, *International journal of multiphase flow* **27**(4): 635–655.

Gumerov, N. A. (1994). *Equations describing the propagation of nonlinear modulation waves in bubbly liquids*, Springer, pp. 131–140.

Hara, S. and Schowalter, W. (1984). Dynamics of nonspherical bubbles surrounded by viscoelastic fluid, *Journal of non-newtonian fluid mechanics* **14**: 249–264.

Harkin, A., Kaper, T. J. and Nadim, A. (2001). Coupled pulsation and translation of two gas bubbles in a liquid, *Journal of Fluid Mechanics* **445**: 377–411.

Hauke, G., Fuster, D. and Dopazo, C. (2007). Dynamics of a single cavitating and reacting bubble, *Physical Review E* **75**(6): 066310.

Hay, T. A., Hamilton, M. F., Ilinskii, Y. A. and Zabolotskaya, E. A. (2008). Coupled pulsation and translation of a gas bubble and rigid particle, *AIP Conference Proceedings*, Vol. 1022, AIP, pp. 209–212.

Herring, C. (1941). *Theory of the pulsations of the gas bubble produced by an underwater explosion*, Columbia Univ., Division of National Defense Research.

Hsu, S. (1978). The application of the poincare-transform to the lotka-volterra model, *Journal of Mathematical Biology* **6**(1): 67–73.

Hua, C. and Johnsen, E. (2013). Nonlinear oscillations following the rayleigh collapse of a gas bubble in a linear viscoelastic (tissue-like) medium, *Physics of Fluids* **25**(8): 083101.

Ichihara, M., Ohkunitani, H., Ida, Y. and Kameda, M. (2004). Dynamics of bubble oscillation and wave propagation in viscoelastic liquids, *Journal of Volcanology and Geothermal Research* **129**(1–3): 37–60.

Ilinskii, Y. A. and Zabolotskaya, E. (1992). Cooperative radiation and scattering of acoustic waves by gas bubbles in liquids, *The Journal of the Acoustical Society of America* **92**(5): 2837–2841.

Ishii, M. and Hibiki, T. (2010). *Thermo-fluid dynamics of two-phase flow*, Springer Science & Business Media.

Jeffrey, A. (2001). *Advanced engineering mathematics*, Elsevier.

Johnson, R. S. (2006). *Singular perturbation theory: mathematical and analytical techniques with applications to engineering*, Springer Science & Business Media.

Jun-Yi, Z. and Xian-Guo, G. (2006). Miura transformation for the td hierarchy, *Chinese Physics Letters* **23**(1): 1.

Kafiabad, H. A. and Sadeghy, K. (2010). Chaotic behavior of a single spherical gas bubble surrounded by a giesekus liquid: A numerical study, *Journal of Non-Newtonian Fluid Mechanics* **165**(13-14): 800–811.

Kako, M. and Rowlands, G. (1976). Two-dimensional stability of ion-acoustic solitons, *Plasma Physics* **18**(3): 165.

Kaya, D. (2004a). An application of the decomposition method for the kdvb equation, *Applied Mathematics and Computation* **152**(1): 279–288.

Kaya, D. (2004b). An application of the decompositionmethod for the two-dimensional kdv-burgers equation, *Computers & Mathematics with Applications* **48**(10-11): 1659–1665.

Kedrinskiy, V. K. (2006). *Hydrodynamics of Explosion: experiments and models*, Springer Science & Business Media, Germany.

Keller, J. B. and Kolodner, I. I. (1956). Damping of underwater explosion bubble oscillations, *Journal of Applied Physics* **27**(10): 1152–1161.

Keller, J. B. and Miksis, M. (1980). Bubble oscillations of large amplitude, *The Journal of the Acoustical Society of America* **68**(2): 628–633.

Kim, C. (1994). Collapse of spherical bubbles in maxwell fluids, *Journal of non-newtonian fluid mechanics* **55**(1): 37–58.

Kirkwood, J. (2018). *Mathematical physics with partial differential equations*, Academic Press.

Koltsova, I. S., Krynskii, L. O., Mikhailov, I. G. and Pokrovskaya, I. E. (1979). Attenuation of ultrasonic-waves in low-viscosity liquids containing gas-bubbles, *Soviet Physics Acoustics-USSR* **25**(5): 409–413.

Kooiman, K., Vos, H. J., Versluis, M. and de Jong, N. (2014). Acoustic behavior of microbubbles and implications for drug delivery, *Advanced drug delivery reviews* **72**: 28–48.

Kubota, A., Kato, H. and Yamaguchi, H. (1992). A new modelling of cavitating flows: a numerical study of unsteady cavitation on a hydrofoil section, *Journal of fluid Mechanics* **240**: 59–96.

Kudryashov, N. A. (2010). A note on the g '/g-expansion method, *Applied Mathematics and Computation* **217**(4): 1755–1758.

Kudryashov, N. A. and Ryabov, P. N. (2014). Exact solutions of one pattern formation model, *Applied Mathematics and Computation* **232**: 1090–1093.

Kudryashov, N. A. and Sinelshchikov, D. I. (2010a). Nonlinear waves in bubbly liquids with consideration for viscosity and heat transfer, *Physics Letters A* **374**(19-20): 2011–2016.

Kudryashov, N. A. and Sinelshchikov, D. I. (2010b). Nonlinear waves in liquids with gas bubbles with account of viscosity and heat transfer, *Fluid Dynamics* **45**(1): 96–112.

Kudryashov, N. A. and Sinelshchikov, D. I. (2012). Equation for the three-dimensional nonlinear waves in liquid with gas bubbles, *Physica Scripta* **85**(2): 025402.

Kudryashov, N. A. and Sinelshchikov, D. I. (2013). An extended equation for the description of nonlinear waves in a liquid with gas bubbles, *Wave Motion* **50**(3): 351–362.

Kudryashov, N. A. and Sinelshchikov, D. I. (2014). Extended models of non-linear waves in liquid with gas bubbles, *International Journal of Non-Linear Mechanics* **63**: 31–38.

Kudryashov, N. A., Sinelshchikov, D. I., Simos, T. and Tsitouras, C. (2016). Analytical solutions of the rayleigh equation for arbitrary polytropic exponent, *AIP Conference Proceedings*, Vol. 1738, AIP Publishing, p. 230010.

Kudryashov, N. A., Sinelshchikov, D. I. and Volkov, A. K. (2015). Extended two dimensional equation for the description of nonlinear waves in gas-liquid mixture, *Applied Mathematics and Computation* **268**: 581–589.

Kudryashov, N. and Teterev, N. (2008). Numerical simulation of acoustic wave propagation via a liquid with gas bubbles, *Journal of the Acoustical Society of America* **123**(5): 3841.

Landau, L. (1987). Fluid mechanics: volume 6 (course of theoretical physics), *EM Lıfshitz, Publisher: Bu1987: Butterworth-Heinemann* .

Lauterborn, W. and Kurz, T. (2010). Physics of bubble oscillations, *Reports on progress in physics* **73**(10): 106501.

Lauterborn, W. and Vogel, A. (1984). Modern optical techniques in fluid mechanics, *Annual Review of Fluid Mechanics* **16**(1): 223–244.

Leblond, H. (2008). The reductive perturbation method and some of its applications, *Journal of Physics B: Atomic, Molecular and Optical Physics* **41**(4): 043001.

Leighton, T. (2004). From seas to surgeries, from babbling brooks to baby scans: The acoustics of gas bubbles in liquids, *International Journal of Modern Physics B* **18**(25): 3267–3314.

Leighton, T. G. (2015). The acoustic bubble: oceanic bubble acoustics and ultrasonic cleaning, *Proceedings of Meetings on Acoustics* **24**(1): 070006.

Li, B., Chen, Y. and Zhang, H. (2002). Explicit exact solutions for new general two-dimensional kdv-type and two-dimensional kdv–burgers-type equations with nonlinear terms of any order, *Journal of Physics A: Mathematical and General* **35**(39): 8253.

Li, S., Zhang, W.-g., Zhao, Y., Li, Z.-m. and Li, X. (2017). Qualitative analysis and approximate damped oscillatory solutions for a kind of nonlinear dispersive-dissipative equation, *Acta Mathematicae Applicatae Sinica, English Series* **33**(1): 1–24.

Lind, S. J. and Phillips, T. N. (2013). Bubble collapse in compressible fluids using a spectral element marker particle method. part 2. viscoelastic fluids, *International Journal for Numerical Methods in Fluids* **71**(9): 1103–1130.

Louisnard, O. (2012). A simple model of ultrasound propagation in a cavitating liquid. part i: Theory, nonlinear attenuation and traveling wave generation, *Ultrasonics sonochemistry* **19**(1): 56–65.

Lugovtsov, A. A. (2009). Propagation of nonlinear waves in an inhomogeneous gas-liquid medium. derivation of wave equations in the korteweg-de vries approximation, *Journal of applied mechanics and technical physics* **50**(2): 327–335.

Lugovtsov, A. A. (2010). Propagation of nonlinear waves in a gas-liquid medium. exact and approximate analytical solutions of wave equations, *Journal of applied mechanics and technical physics* **51**(1): 44–50.

Luo, J., Fang, Z., Smith, R. L. and Qi, X. (2015). Fundamentals of acoustic cavitation in sonochemistry, *Production of Biofuels and Chemicals with Ultrasound*, Springer, pp. 3–33.

Macpherson, J. D. (1957). The effect of gas bubbles on sound propagation in water, *Proceedings of the Physical Society. Section B* **70**(1): 85.

Malfliet, W. (1992). Solitary wave solutions of nonlinear wave equations, *American Journal of Physics* **60**(7): 650–654.

Mettin, R., Akhatov, I., Parlitz, U., Ohl, C. D. and Lauterborn, W. (1997). Bjerknes forces between small cavitation bubbles in a strong acoustic field, *Physical review E* **56**(3): 2924.

Miksis, M. J. and Ting, L. (1992a). Effective equations for multiphase flows-waves in a bubbly liquid, *Advances in Applied Mechanics* **28**: 141–260.

Miksis, M. J. and Ting, L. (1992b). Wave propagation in a bubbly liquid at small volume fraction, *Chemical Engineering Communications* **118**(1): 59–73.

Minnaert, M. (1933). Xvi. on musical air-bubbles and the sounds of running water, *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science* **16**(104): 235–248.

Mirzazadeh, M. and Eslami, M. (2012). Exact solutions of the kudryashov-sinelshchikov equation and nonlinear telegraph equation via the first integral method, *Nonlinear Anal. Model. Control* **17**(4): 481–488.

Mond, M. (1987). Kinetic theory of waves in bubbly liquids, *Physics Letters A* **122**(3): 153–156.

Morrison, F. A. (2013). *An introduction to fluid mechanics*, Cambridge University Press.

Munteanu, L. and Donescu, S. (2004). *Introduction to soliton theory: applications to mechanics*, Vol. 143, Springer Science & Business Media.

Nakoryakov, V. E., Pokusaev, B. G. and Shreiber, I. R. (1993). *Wave propagation in gas-liquid media*, CRC Press.

Neppiras, E. A. and Noltingk, B. E. (1951). Cavitation produced by ultrasonics: Theoretical conditions for the onset of cavitation, *Proceedings of the Physical Society. Section B* **64**(12): 1032.

Nigmatulin, R. I. (1982). Mathematical modelling of bubbly liquid motion and hydrodynamical effects in wave propagation phenomenon, *Mechanics and Physics of Bubbles in Liquids*, Springer, pp. 267–289.

Nigmatulin, R. I. (1990). *Dynamics of multiphase media*, Vol. 2, CRC Press.

Nigmatulin, R. I. and Khabeev, N. S. (1974). Heat exchange between a gas bubble and a liquid, *Fluid Dynamics* **9**(5): 759–764.

Nigmatulin, R. I. and Shagapov, V. S. (1974). Structure of shock waves in a liquid containing gas bubbles, *Fluid Dynamics* **9**(6): 890–899.

Noltingk, B. E. and Neppiras, E. A. (1950). Cavitation produced by ultrasonics, *Proceedings of the Physical Society. Section B* **63**(9): 674.

Oguz, H. N. and Prosperetti, A. (1990). A generalization of the impulse and virial theorems with an application to bubble oscillations, *Journal of Fluid Mechanics* **218**: 143–162.

Oldroyd, J. (1950). On the formulation of rheological equations of state, *Proceedings of the Royal Society of London. Series A. Mathematical and Physical Sciences* **200**(1063): 523–541.

Oliveri, F. (1989). Nonlinear wave propagation in a non-diffusive model of bubbly liquids, *Acta Mechanica* **83**(3): 135–148.

Oskolkov, A. (1985). Theory of nonstationary flows of kelvin-voigt fluids, *Journal of Soviet Mathematics* **28**(5): 751–758.

Pal, R. (2007). Steady laminar flow of non-newtonian bubbly suspensions in pipes, *Journal of Non-Newtonian Fluid Mechanics* **147**(1-2): 129–137.

Pang, M., Wei, J. and Yu, B. (2010). Numerical study of bubbly upflows in a vertical channel using the euler–lagrange two-way model, *Chemical Engineering Science* **65**(23): 6215–6228.

Perko, L. (2013). *Differential equations and dynamical systems*, Vol. 7, Springer Science & Business Media.

Plesset, M. S. and Hsieh, D.-Y. (1960). Theory of gas bubble dynamics in oscillating pressure fields, *The Physics of Fluids* **3**(6): 882–892.

Plesset, M. S. and Prosperetti, A. (1977). Bubble dynamics and cavitation, *Annual review of fluid mechanics* **9**(1): 145–185.

Prosperetti, A. (1977). Thermal effects and damping mechanisms in the forced radial oscillations of gas bubbles in liquids, *The Journal of the Acoustical Society of America* **61**(1): 17–27.

Prosperetti, A. (1982). Bubble dynamics: a review and some recent results, *Applied Scientific Research* **38**(1): 145–164.

Prosperetti, A. (1984). Bubble phenomena in sound fields: part one, *Ultrasonics* **22**(2): 69–77.

Prosperetti, A. (1987). The equation of bubble dynamics in a compressible liquid, *The Physics of fluids* **30**(11): 3626–3628.

Prosperetti, A. (2004). Bubbles, *Physics of fluids* **16**(6): 1852–1865.

Prosperetti, A. (2006). The thermal behaviour of oscillating gas bubbles, *Journal of Fluid Mechanics* **222**: 587–616.

Prosperetti, A., Crum, L. A. and Commander, K. W. (1988). Nonlinear bubble dynamics, *The Journal of the Acoustical Society of America* **83**(2): 502–514.

Prosperetti, A. and Lezzi, A. (1986). Bubble dynamics in a compressible liquid. part 1. first-order theory, *Journal of Fluid Mechanics* **168**: 457–478.

Puente, G. F. and Bonetto, F. J. (2005). Proposed method to estimate the liquid-vapor accommodation coefficient based on experimental sonoluminescence data, *Physical Review E* **71**(5): 056309.

Qin, Y., Wang, Z., Zou, L. and He, M. (2018). Parametric analytical solution for the n-dimensional rayleigh equation, *Applied Mathematics Letters* **76**: 8–13.

Rady, A. A., Osman, E. and Khalfallah, M. (2010). The homogeneous balance method and its application to the benjamin–bona–mahoney (bbm) equation, *Applied mathematics and computation* **217**(4): 1385–1390.

Rayleigh, L. (1917). Viii. on the pressure developed in a liquid during the collapse of a spherical cavity, *Philosophical Magazine Series 6* **34**(200): 94–98.

Rozenberg, L. (2013). *High-intensity ultrasonic fields*, Springer Science & Business Media, Berlin.

Ru-Guang, Z. and Jie, C. (2015). Two hierarchies of new differential-difference equations related to the darboux transformations of the kaup—newell hierarchy, *Communications in Theoretical Physics* **63**(1): 1.

Ruggles, A. E., Lahey, R. T., Drew, D. A. and Scarton, H. A. (1988). An investigation of the propagation of pressure perturbations in bubbly air/water flows, *Journal of heat transfer* **110**(2): 494–499.

Rui, W. and Qi, X. (2016). Bilinear approach to quasi-periodic wave solutions of the kersten-krasil'shchik coupled kdv-mkdv system, *Boundary Value Problems* **2016**(1): 130.

Ryskin, G. (1990). Dynamics and sound emission of a spherical cavitation bubble in a dilute polymer solution, *Journal of Fluid Mechanics* **218**: 239–263.

Sajid, M., Hayat, T. and Asghar, S. (2006). On the analytic solution of the steady flow of a fourth grade fluid, *Physics Letters A* **355**(1): 18–26.

Salas, A. H. and Gómez, C. A. (2008). Computing exact solutions for some fifth kdv equations with forcing term, *Applied Mathematics and Computation* **204**(1): 257–260.

Salas, S., Alvaro, H., Gomez, S. and Cesar, A. (2009). Exact solutions for a third-order kdv equation with variable coefficients and forcing term, *Mathematical Problems in Engineering* **2009**.

Scardina, P. and Edwards, M. (2001). Prediction and measurement of bubble formation in water treatment, *Journal of Environmental Engineering* **127**(11): 968–973.

Seo, J. H., Lele, S. K. and Tryggvason, G. (2010). Investigation and modeling of bubble-bubble interaction effect in homogeneous bubbly flows, *Physics of Fluids* **22**(6): 063302.

Shah, A. and Saeed, R. (2009). Ion acoustic shock waves in a relativistic electron–positron–ion plasmas, *Physics Letters A* **373**(45): 4164–4168.

Shima, A., Tsujino, T. and Nanjo, H. (1986). Nonlinear oscillations of gas bubbles in viscoelastic fluids, *Ultrasonics* **24**(3): 142–147.

Shklyaev, S. and Straube, A. V. (2010). Formation and evolution of bubbly screens in confined oscillating bubbly liquids, *Physical Review E* **81**(1): 016321.

Silberman, E. (1957). Sound velocity and attenuation in bubbly mixtures measured in standing wave tubes, *The journal of the Acoustical Society of America* **29**(8): 925–933.

Singh, S. and Ray, S. S. (2017). Exact solutions for the wick-type stochastic kersten-krasil'shchik coupled kdv-mkdv equations, *The European Physical Journal Plus* **132**(11): 480.

Storey, B. D. and Szeri, A. J. (1999). Mixture segregation within sonoluminescence bubbles, *Journal of Fluid Mechanics* **396**: 203–221.

Storey, B. D. and Szeri, A. J. (2000). Water vapour, sonoluminescence and sonochemistry, *Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences*, Vol. 456, The Royal Society, pp. 1685–1709.

Stricker, L., Prosperetti, A. and Lohse, D. (2011). Validation of an approximate model for the thermal behavior in acoustically driven bubbles, *The Journal of the Acoustical Society of America* **130**(5): 3243–3251.

Sun, Y., Tian, B., Wu, X.-Y., Liu, L. and Yuan, Y.-Q. (2017). Dark solitons for a variable-coefficient higher-order nonlinear schrödinger equation in the inhomogeneous optical fiber, *Modern Physics Letters B* **31**(12): 1750065.

Tanasawa, I. and Yang, W. (1970). Dynamic behavior of a gas bubble in viscoelastic liquids, *Journal of Applied Physics* **41**(11): 4526–4531.

Towne, D. H. (2014). *Wave phenomena*, Courier Dover Publications.

Trilling, L. (1952). The collapse and rebound of a gas bubble, *Journal of Applied Physics* **23**(1): 14–17.

Wang, Q. and Blake, J. (2010). Non-spherical bubble dynamics in a compressible liquid. part 1. travelling acoustic wave, *Journal of Fluid Mechanics* **659**: 191–224.

Wang, Q. and Blake, J. (2011). Non-spherical bubble dynamics in a compressible liquid. part 2. acoustic standing wave, *Journal of Fluid Mechanics* **679**: 559–581.

Wang, T.-Y., Ren, Y.-H. and Zhao, Y.-L. (2006). Exact solutions of (3 + 1)-dimensional stochastic burgers equation, *Chaos, Solitons & Fractals* **29**(4): 920–927.

Wang, Y.-C. and Brennen, C. E. (1999). Numerical computation of shock waves in a spherical cloud of cavitation bubbles, *Journal of fluids engineering* **121**(4): 872–880.

Wang, Z., Qin, Y. and Zou, L. (2018). A kind of analytical solution for the rayleigh-plesset equation in n-dimensions, *AIP Conference Proceedings*, Vol. 1978, AIP Publishing, p. 470069.

Warnez, M. and Johnsen, E. (2015). Numerical modeling of bubble dynamics in viscoelastic media with relaxation, *Physics of Fluids* **27**(6): 063103.

Washimi, H. and Taniuti, T. (1966). Propagation of ion-acoustic solitary waves of small amplitude, *Physical Review Letters* **17**(19): 996.

Watanabe, M. and Prosperetti, A. (1994). Shock waves in dilute bubbly liquids, *Journal of fluid mechanics* **274**: 349–381.

Wazwaz, A.-M. (2001). A new algorithm for solving differential equations of lane–emden type, *Applied Mathematics and Computation* **118**(2-3): 287–310.

Wazwaz, A.-M. (2006). The modified decomposition method for analytic treatment of differential equations, *Applied Mathematics and Computation* **173**(1): 165–176.

Wazwaz, A.-M. (2010a). *Partial differential equations and solitary waves theory*, Springer Science & Business Media.

Wazwaz, A.-M. (2010b). A study on kdv and gardner equations with time-dependent coefficients and forcing terms, *Applied Mathematics and Computation* **217**(5): 2277–2281.

Wazwaz, A.-M. and Mehanna, M. S. (2010). A variety of exact travelling wave solutions for the (2 + 1)-dimensional boiti–leon–pempinelli equation, *Applied Mathematics and Computation* **217**(4): 1484–1490.

Wazzan, L. (2009). A modified tanh–coth method for solving the kdv and the kdv–burgers' equations, *Communications in Nonlinear Science and Numerical Simulation* **14**(2): 443–450.

Wijngaarden, L. V. (1968). On the equations of motion for mixtures of liquid and gas bubbles, *Journal of Fluid Mechanics* **33**(03): 465–474.

Wijngaarden, L. V. (1972). One-dimensional flow of liquids containing small gas bubbles, *Annual review of fluid Mechanics* **4**(1): 369–396.

Wolf, A., Rauh, C. and Delgado, A. (2016). Dynamics and long-time behavior of a small bubble in viscous liquids with applications to food rheology, *Archive of Applied Mechanics* **86**(6): 979–1002.

Xu, M.-J., Tian, S.-F., Tu, J.-M. and Zhang, T.-T. (2016). Bäcklund transformation, infinite conservation laws and periodic wave solutions to a generalized (2+ 1)-dimensional boussinesq equation, *Nonlinear Analysis: Real World Applications* **31**: 388–408.

Xu, Y. and Shu, C.-w. (2004). Local discontinuous galerkin methods for three classes of nonlinear wave equations, *Journal of Computational Mathematics* pp. 250–274.

Yang, D., Ni, Z., Yang, Y., Xu, G., Tu, J., Guo, X., Huang, P. and Zhang, D. (2018). The enhanced hifu-induced thermal effect via magnetic ultrasound contrast agent microbubbles, *Ultrasonics sonochemistry* **49**: 111–117.

Yang, X. and Church, C. C. (2005a). A model for the dynamics of gas bubbles in soft tissue, *The Journal of the Acoustical Society of America* **118**: 3595.

Yang, X. and Church, C. C. (2005b). Nonlinear dynamics of gas bubbles in viscoelastic media, *Acoustics Research Letters Online* **6**(3): 151–156.

Zabolotskaya, E. A. (1984). Interaction of gas-bubbles in a sound field, *Soviet Physics Acoustics-USSR* **30**(5): 365–368.

Zahibo, N., Pelinovsky, E. and Sergeeva, A. (2009). Weakly damped kdv soliton dynamics with the random force, *Chaos, Solitons & Fractals* **39**(4): 1645–1650.

Zaki, S. (2000). Solitary waves of the korteweg–de vries–burgers' equation, *Computer physics communications* **126**(3): 207–218.

Zedan, H. A. and El Adrous, E. (2012). The application of the homotopy perturbation method and the homotopy analysis method to the generalized zakharov equations, *Abstract and Applied Analysis*, Vol. 2012, Hindawi.

Zhang, D. Z. and Prosperetti, A. (1994a). Averaged equations for inviscid disperse two-phase flow, *Journal of Fluid Mechanics* **267**: 185–219.

Zhang, D. Z. and Prosperetti, A. (1994b). Ensemble phase-averaged equations for bubbly flows, *Physics of Fluids* **6**(9): 2956–2970.

Zhi-Fen, Z., Tong-Ren, D., Wen-Zao, H. and Zhen-Xi, D. (2006). *Qualitative Theory of differential equations*, Vol. 101, American Mathematical Soc.

Zhong, P. (2013). Shock wave lithotripsy, *Bubble dynamics and shock waves*, Springer, pp. 291–338.

LIST OF PUBLICATIONS

ISI Journals

1. **Chukkol, Y. B., Mukhiddin, M.**, (2020). Shock Waves Propagation in an Isothermal Slightly Compressible Bubbly Viscoelastic Fluid Flow. *Modern Physics Letters B* Vol 34, No 3, pp 2050042-1 - 2050042-17.

2. **Chukkol, Y. B., Mukhiddin, M.**, (2020). Bright and Dark Soliton Solutions of Three-Dimensional KdV and mKdV Equations in an Inviscid Liquid with Gas Bubbles. *International Journal of Mathematics and Computer Science* Vol 15, No 1, pp 253?264.

3. **Chukkol, Y. B., Mukhiddin, M.**, (2020). Kink Wave Solutions to KdV-Burgers equation with Forcing Term. *Communications of the Korean Mathematical Society* Vol 35, No 2, pp 685-695.

4. **Chukkol, Y. B., Mohamad, N. B. M., Mukhiddin, M.**, (2019). A Complex Solutions to (3+1)-Dimensional KdV and (3+1)-Dimensional KdV-Burger Equations in Bubbly Liquid. *Journal of Interdisciplinary Mathematics* Vol 22, No 3, pp 773-785.

Scopus Journal

1. **Chukkol, Y. B., Mohamad, N. B. M., Mukhiddin, M.** (2018). Explicit solutions to the (3+1)-dimensional Kudryashov-Sinelshchikov equations in bubbly flow dynamics. *Journal of Applied Mathematics*, Volume 2018, 2018, Article number 7452786.

Indexed Conference Proceedings

1. **Chukkol, Y. B., Mohamad, N. B. M., Mukhiddin, M.** (2019). Shock wave propagation in bubbly viscoelastic fluid flow. AIP Conference Proceedings 2184, 060058; https://doi.org/10.1063/1.5136490 (Malaysia)

2. **Chukkol, Y. B., Mohamad, N. B. M., Mukhiddin, M.** (2017). Exact solutions to the KDV-Burgers equation with forcing term using Tanh-Coth method. AIP Conference Proceedings 1870, 040024 (2017); doi: 10.1063/1.4995856.