

IDENTIFICATION OF THE UNIQUE SPHEROID BY ROTATED AND SCALED
FIRST ORDER POLARIZATION TENSOR

SYAFINA BINTI AHMAD

UNIVERSITI TEKNOLOGI MALAYSIA

IDENTIFICATION OF THE UNIQUE SPHEROID BY ROTATED AND SCALED
FIRST ORDER POLARIZATION TENSOR

SYAFINA BINTI AHMAD

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ABSTRACT

In the applications of electric and electromagnetic, polarization tensor is usually used to describe the perturbation in electrical field induced by the presence of conducting objects. Due to the fact that polarization tensor carries information about geometry and conductivity of those presented conducting objects, it is possible to substantially use the polarization tensor to describe the objects including reconstructing their images. Some recent applications of polarization tensor include electrical imaging for medical or industrial purposes, characterization of objects by weakly electrosensing fish and also metal detection. Specifically, this research is concerned with the first order polarization tensor when the conducting object is a conducting spheroid. Given the first order polarization tensor of any object, a spheroid can be numerically determined so that the spheroid has the same first order polarization tensor. Here, the main purpose of this research is to investigate the rotated or the scaled first order polarization tensor that is related to a unique spheroid. In order to achieve the objectives of this research, the depolarization factors of both prolate and oblate spheroids are first investigated. It is proven that the eccentricity of both spheroids are unique based on their depolarization factors. After that, the effect of scaling on the first order polarization tensor to the volume, depolarization factors, eccentricity and also semi axes of the spheroid are revealed. Some numerical calculations to identify the volume and semi axes of the spheroid based on a few scaled first order polarization tensors are presented in this research. Furthermore, the effect of rotation on the first order polarization tensor to the conductivity, volume, depolarization factors as well as semi axes of the spheroid are also identified. By the implementation of rotation, a new and improvised flowchart is provided in finding the semi axes of a spheroid based on the different forms of the first order polarization tensor. Last but not least, this research described the uniqueness of the spheroid at a fixed conductivity based on either rotated or scaled first order polarization tensor. As a conclusion, this research would be beneficial in gaining the information about the object specifically for spheroid based on the given first order polarization tensor.

ABSTRAK

Dalam aplikasi elektrik dan elektromagnetik, tensor pengutuban biasanya digunakan untuk menerangkan gangguan medan elektrik yang disebabkan oleh kehadiran objek yang mengkonduksikan elektrik. Disebabkan oleh fakta bahawa tensor pengutuban mengandungi maklumat mengenai geometri dan kekonduksian objek yang hadir, adalah mungkin untuk menggunakan tensor pengutuban dalam mencirikan objek berkenaan termasuk pembentukan semula imej objek tersebut. Beberapa aplikasi terkini tensor pengutuban termasuklah pengimejan elektrik untuk tujuan perubatan atau industri, pencirian objek oleh ikan berderia elektrik lemah dan juga pengesanan logam. Secara khususnya, penyelidikan ini adalah berkenaan dengan tensor pengutuban peringkat pertama apabila objek yang mengkonduksikan elektrik tersebut adalah sferoid. Diberi tensor pengutuban peringkat pertama untuk sebarang objek, sebuah sferoid yang mempunyai tensor pengutuban peringkat pertama yang sama dapat ditentukan dengan kaedah berangka. Di sini, tujuan utama penyelidikan ini adalah untuk mengkaji tensor pengutuban peringkat pertama yang diputar dan berskala bagi sesebuah sferoid yang unik. Bagi mencapai semua objektif penyelidikan ini, faktor penyahkutuban bagi kedua-dua sferoid lonjong dan sferoid buntal terlebih dahulu dikaji. Terbukti bahawa keeksentrikan kedua-dua sferoid adalah unik berdasarkan faktor penyahkutuban mereka. Kemudian, kesan penskalaan pada tensor pengutuban peringkat pertama terhadap isipadu, faktor penyahkutuban, keeksentrikan dan juga paksi separa sferoid terbabit didedahkan. Beberapa pengiraan berangka untuk menentukan isipadu dan paksi separa bagi sesebuah sferoid berdasarkan tensor pengutuban peringkat pertama berskala dinyatakan dalam kajian ini. Tambahan pula, kesan putaran tensor pengutuban peringkat pertama terhadap kekonduksian, isipadu, faktor penyahkutuban dan juga paksi separa bagi sesebuah spheroid dikenalpasti. Berdasarkan putaran tensor pengutuban, carta alir kaedah yang terbaharu disediakan bagi mencari semi paksi spheroid berdasarkan pelbagai bentuk tensor pengutuban peringkat pertama. Akhir sekali, kajian ini juga menerangkan tentang keunikan sferoid apabila kekonduksian ditetapkan berdasarkan tensor pengutuban peringkat pertama berskala atau yang diputar. Sebagai kesimpulan, penyelidikan ini sangat berfaedah dalam mendapatkan maklumat mengenai objek khususnya sferoid berdasarkan tensor pengutuban peringkat pertama yang telah ditentukan terlebih dahulu.

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LIST OF ABBREVIATIONS

PT	-	Polarization Tensor
GPT	-	Generalized Polarization Tensor
FEM	-	Finite Element Method
BEM	-	Boundary Element Method
EIT	-	Electrical Impedance Tomography

LIST OF SYMBOLS

$>$	-	Greater than
$<$	-	Less than
$=$	-	Equal
\mathbb{R}	-	Real number
\neq	-	Not equal
∞	-	Infinity
ψ	-	Psi
φ	-	Varphi
\in	-	Element of
θ	-	Theta
\Leftrightarrow	-	If and only if

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CHAPTER 1

INTRODUCTION

1.1 Preliminaries

Polarization tensor (PT) has been widely used in the studies of engineering and science to represent the perturbation of electric field caused by the presence of conducting objects. This circumstance describes the process of object identification and characterization by implementing the PT terminology. Recent research on the PT focuses mainly on improving image reconstruction in electrical imaging for industrial or biomedical uses. It is shown that the perturbation of electric field can be expressed in term of an asymptotic expansion and it is the PT called as the Generalized Polarization Tensor (GPT) [1].

Moreover, PT can be specified as the PT of an object which interprets the perturbation with the presence of that object. In addition, the first order PT contains information about the shape of the objects and its conductivity. By using the explicit formula as given in [1], the PT that represents the related disturbance of a conducting object can be determined when the geometry and conductivity of the object are fixed. This method of finding the PT has a lower computational cost for describing the object when it is not compulsory to find the full image of the object.

A concept known as the depolarization factors also has a relationship with the first order PT (the simplest form of GPT). The depolarization factors had previously been used in studying magnetic problems. There are studies that discuss depolarization factors specifically for an ellipsoid in [2, 3, 4]. In these research, various explicit formulas for the depolarization factors of ellipsoid mainly for prolate and oblate spheroid are presented. In PT, it is possible to simplify the explicit formula of the first order PT for conducting ellipsoids using these formulas.

1.2 Research Background

In this research, the first order GPT (or the first order PT) that represents the perturbation of electric field because of the existence of conducting spheroid or two semi equal ellipsoid (see Figure 1.1 and Figure 1.2) is studied. Despite the fact that previous studies have shown that the first order PT for many objects is actually related to the first order PT for ellipsoid, this research highlights on spheroid to describe and determine a few spheroids that are represented by the given first order PT. Nonetheless, this research focuses on investigating the effect of scaled and rotated first order PT to the physical properties of a spheroid that is represented by the given first order PT. Several properties of the first order PT for spheroid that have been defined and proved in [5] will be used in order to implement the transformations of the first order PT.

The research on the rotation and scaling of the first order PT is important in order to characterize the unique object represented by the PT itself. There are some researches that have been conducted to investigate and determine the effect of transformations of the object to its first order PT such as in [6, 7, 8]. Therefore, it is necessary to understand the transformations on the first order PT and hence determine the unique object so that it will be useful to characterize and identify the object.

Moreover, Figure 1.3 shows the flow of the research on how the rotated and scaled first order PT is used in some applications such as metal detector and electrosensing fish. When the electric field is generated in a space and the object (or spheroid) passed the field, the perturbation of the electric field is occurred. The perturbation is then transformed into first order PT which contains information of the spheroid either it is rotated or scaled. Lastly, the properties of the spheroid can be identified from the rotated and scaled first order PT. As the spheroid is identified, it might be important to determine that the spheroid is unique for a fixed conductivity.

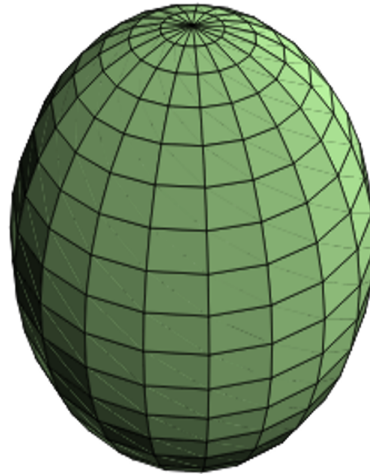


Figure 1.1: Prolate spheroid

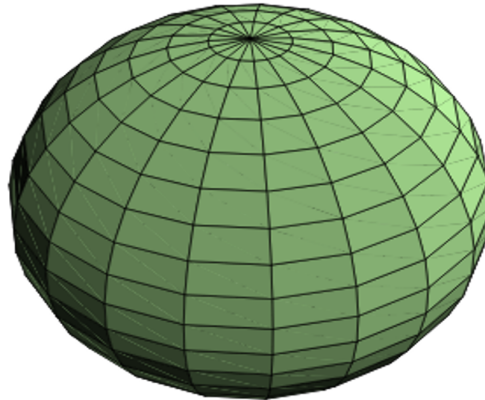


Figure 1.2: Oblate spheroid

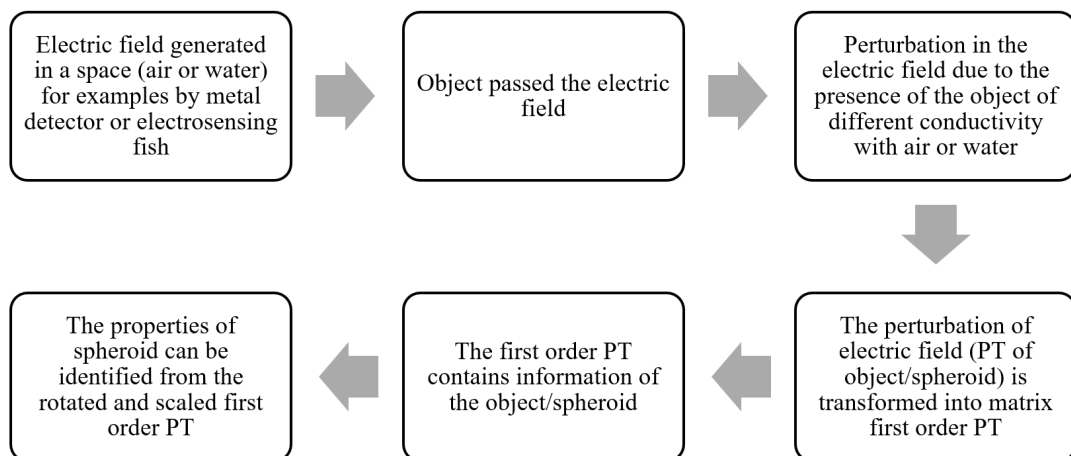


Figure 1.3: The flow of research on how the polarization tensor works in some applications

1.3 Problem Statement

The first order PT involved the understanding in geometry and physical properties of the objects. Nonetheless, there are still lack of research regarding the implementation of rotated and scaled first order PT in describing the object since most of the study only focused on the object when the object is rotated and scaled. Based on the research conducted by [5], there is a limitation on the algorithm where the method for finding the semi axes of spheroid is considered for one case only and did not implement rotation of the first order PT. Moreover, the research studied by [9] is also lacked on the theoretical results although the algorithm is considered for all cases of the first order PT but did not consider a fixed conductivity as stated in [5]. Therefore, this research will further investigate the effect of rotation and scaling on the first order PT of the spheroid as well as its theoretical results.

On top of that, the research on the uniqueness of spheroid is still insufficient based on the first order PT because the computation of the first order PT and the semi axes for the ellipsoid are the main focus by the previous study. Consequently, the approaches on the theory of characterizing the object based on its PT are necessary. By the implementation of mathematical analysis on the uniqueness of spheroid, it will be a lot useful in order to describe and justify the objects. Therefore, this research will prove the uniqueness theorems for object, specifically a spheroid, based on the first order PT.

1.4 Objectives of the Research

This research will study on the first order PT of spheroid. The objectives of this research can be summarized as follows:

- i. to show the uniqueness of the eccentricity of a spheroid based on the depolarization factors.

- ii. to investigate the effect of scaling and rotation on the first order PT for a spheroid.
- iii. to determine the volume, depolarization factors, eccentricity and also semi axes of spheroid from a scaled and rotated first order PT.
- iv. to describe the uniqueness of the volume and the depolarization factor of a spheroid based on a given first order PT.
- v. to explain the uniqueness of the semi axes of a spheroid for a given first order PT at a fixed conductivity.

1.5 Scope of the Research

This research focuses on the proving of the properties of spheroid with the application of rotation and scaling in order to determine all the semi principal axes of the prolate spheroid ($a > b = c, b > a = c, c > a = b$) and oblate spheroid ($a < b = c, b < a = c, c < a = b$) based on the given first order PT. These properties include conductivity, volume, depolarization factors and also semi axes of the spheroid. Apart from that, the uniqueness of spheroid which involves volume, depolarization factors, eccentricity and semi axes based on the first order PT will be discussed in this research.

1.6 Significance of the Research

This research gives an important contribution as it would be beneficial for the researchers to gain new knowledge on characterizing the spheroid when its first order PT is rotated and scaled for any related applications. Based on the given first order PT, the size of any objects can be predicted. In spite of that, this study can also be a useful guideline in the future research for the computation and the uniqueness of spheroid for all cases such that when the semi axes are $a > b = c, a < b = c, b > a = c, b < a = c, c > a = b$ and $c < a = b$ according to the first order PT.

1.7 Thesis Outline

This thesis consists of seven chapters. The research framework of this study can be clarified and described as in Figure 1.4 below.

Firstly, Chapter 1 discusses on the introduction of this study. This chapter gives a brief introduction of Polarization Tensor (PT) which includes the discussion on the research background and problem statement. Besides, the objectives of the research, scope of the research and significance of the research are also stated. Finally, the overall research framework is explained.

Next, chapter 2 presents the literature review of this research. The mathematical formulation of the first order PT as well as its properties are stated in this chapter. The properties of depolarization factors are also reviewed. Apart from that, the transformations of the first order PT including rotation and scaling are explained in this chapter. The method on obtaining the semi axes of spheroid based on first order PT is discussed. Lastly, some mathematical analysis are also presented.

Chapter 3 begins by investigating on some properties of the depolarization factors of both prolate and oblate spheroids. The unique solution of the eccentricity with respect to the depolarization factors are also discussed in this chapter. The proofs are stated in details.

Then, the effect of scaling on the semi axes of an ellipsoid and the first order PT are discussed in Chapter 4. Considering the effect on scaling the first order PT, it is possible to determine the semi axes of the spheroids. The numerical examples are discussed in this chapter as well as the effect of computation to the volume and semi axes based on the size of scaling factor.

On the other hand, Chapter 5 discusses on the effect of rotation of the first order PT to the conductivity, volume and depolarization factor of the spheroid. Moreover, a new flow chart on computing the semi axes of both spheroids for all cases is also provided which is an updated from the previous research.

Furthermore, Chapter 6 focuses on the uniqueness of volume and depolarization factor for all cases of the first order PT of spheroid. Besides, the uniqueness of semi axes is also investigated based on the first order PT of the spheroid.

The summary of the entire thesis is presented in the last chapter, Chapter 7. Moreover, some suggestions on the further research that can be conducted are given within the chapter. In addition, Figure 1.5 summarizes the organization of the thesis.

1.8 Conclusion

This chapter has summarized the entire research for this thesis. In addition, the research background, problem statement, research objectives, scope of the research and significance of research are also stated here. Besides, this chapter also includes a thesis outline that explains the entire thesis.

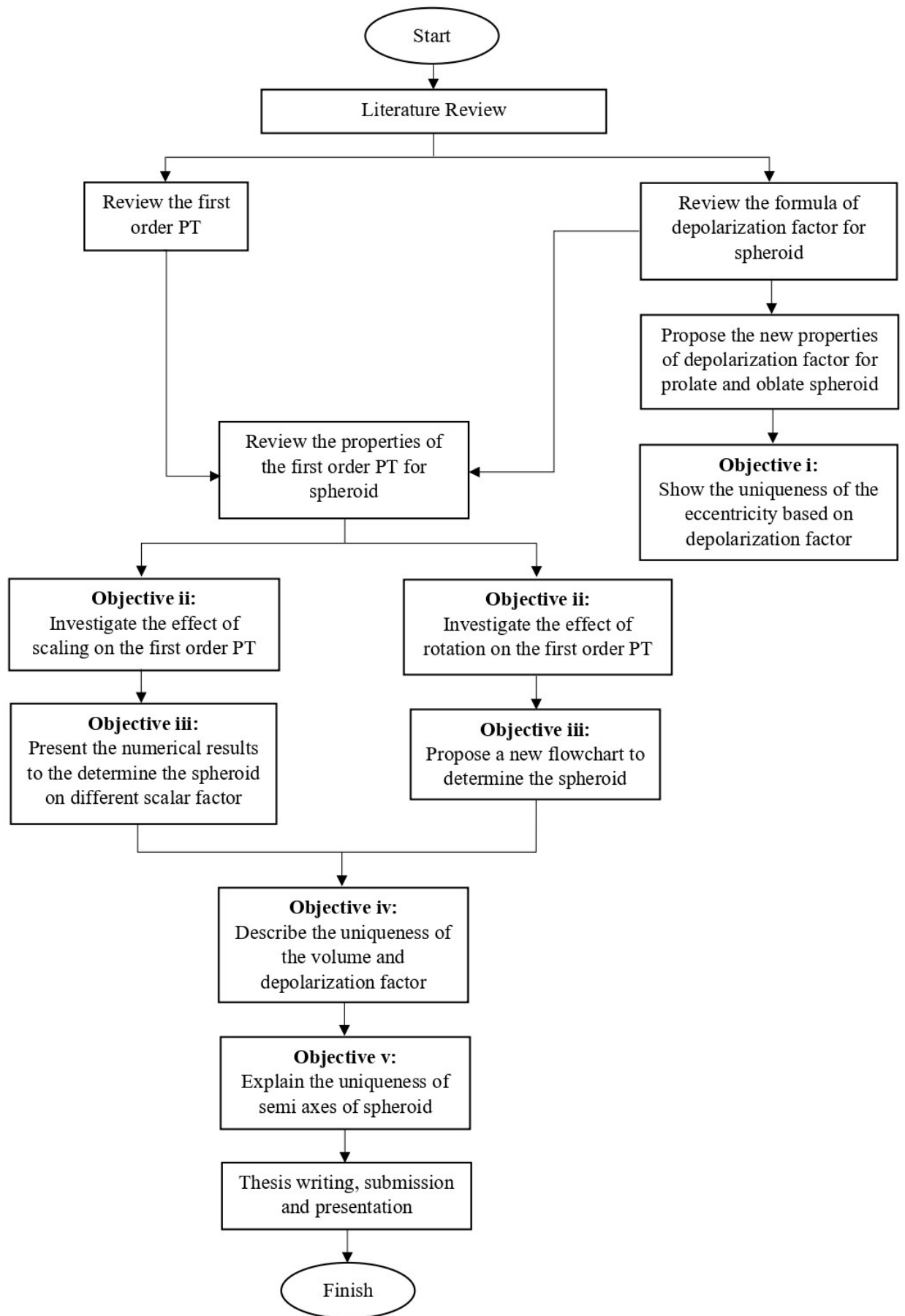


Figure 1.4: Research Framework

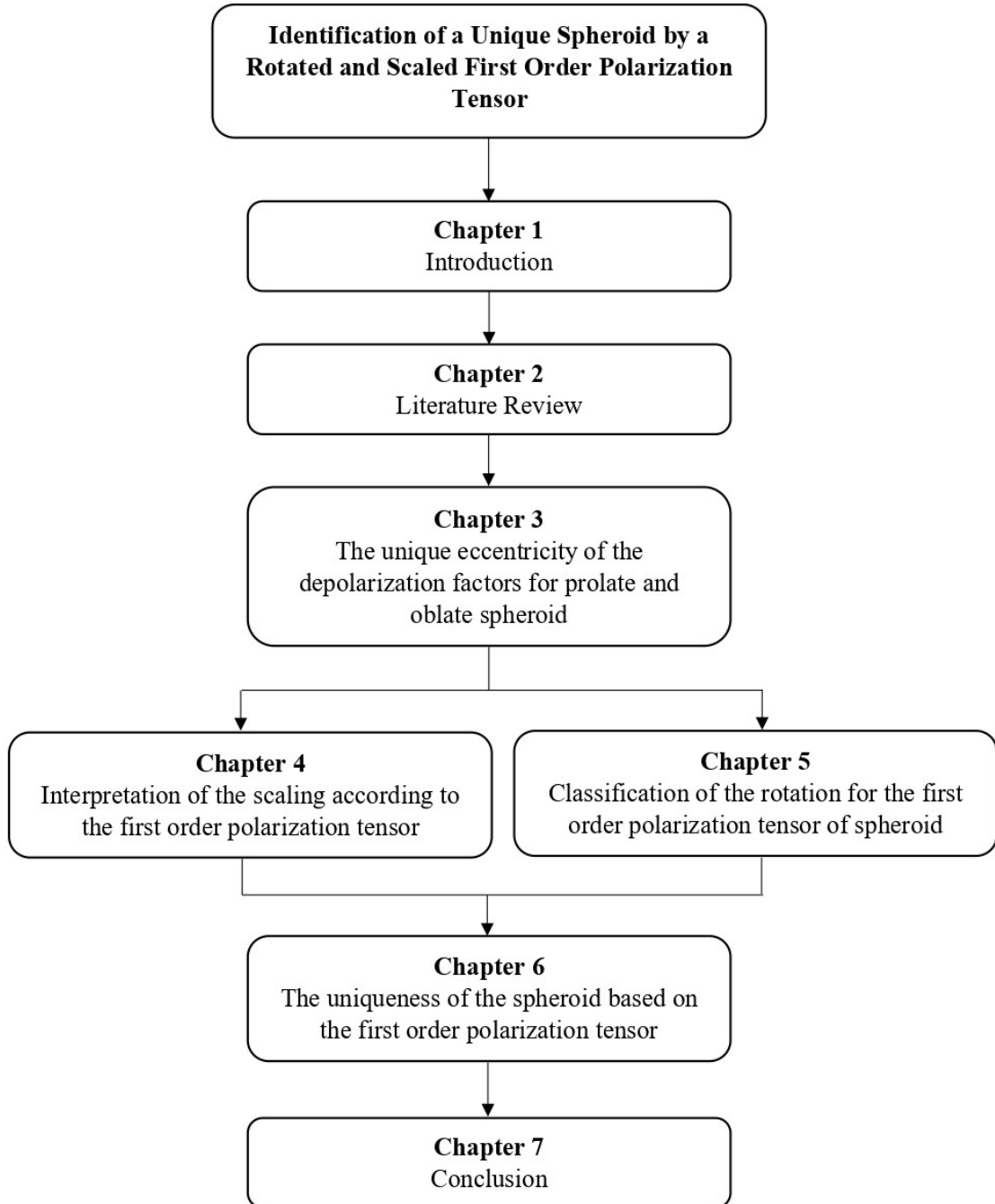


Figure 1.5: Thesis Organization

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LIST OF PUBLICATIONS

Indexed Journal

1. Ahmed Sukri, S. A., Khairuddin, T. K. A., Muminov, M., Hoe, Y. S., and **Ahmad, S.** The Effect of Different Scale on Object to the Approximation of the First Order Polarization Tensor of Sphere, Ellipsoid, and Cube. *Journal of Advanced Research in Fluid Mechanics and Thermal Sciences*. 87(1): 108-117. 2021.
2. **Ahmad, S.**, Yunos, N. M., Khairuddin, T. K. A., Muminov, M., and Sukri, S. A. The Effect on The Volume and Semi Axes of a Conducting Spheroid Due to The Scaling on Its First Order Polarization Tensor. *Journal of Advanced Research in Fluid Mechanics and Thermal Sciences*. 94(1): 1-18. 2022.

Non-Indexed Journal

1. Bahuriddin, N. S., Muminov, M., Khairuddin, T. K. A., **Ahmad, S.** and Ibrahim, W. R. W. An Extended Method for Fitting the First Order Polarization Tensor to a Spheroid. *Journal of Advanced Research in Applied Sciences and Engineering Technology*. 23(1): 8–17. 2021.

Indexed Conference Proceedings

1. **Ahmad, S.**, Yunos, N. M., Khairuddin, T. K. A., and Embong, A. F. The Unique Eccentricity of a Prolate Spheroid Based on its Depolarization Factor. *American Institute of Physics (AIP) Conference Proceedings series*. Submitted.