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Adaptive Memory Control Charts Constructed on Generalized Likelihood Ratio Test to Monitor Process Location

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Abstract
An adaptive cumulative sum (CUSUM) control chart based on the classical exponential weighted moving average (EWMA) statistic and Huber's function, symbolized as an $\mathrm{ACUSUM_E}$ control chart, is an enhanced form of the classical CUSUM control chart that can identify different sizes of shift. However, the classical EWMA statistic for the $\mathrm{ACUSUM_E}$ control chart does not provide explicit rule for parameter choices to diagnose a specific shift. To overcome this issue, this study has proposed two ACUSUM control charts, symbolized as $\mathrm{ACUSUM_c}$ control charts to monitor a specific and a certain range of shift. The novelty behind the proposed $\mathrm{ACUSUM_c}$ ($\mathrm{ACUSUM_c^{(1)}}$ and $\mathrm{ACUSUM_c^{(2)}}$) control charts is initially adaptively updating the reference parameter using the classical CUSUM statistic, generalized likelihood ratio test, and score functions to achieve superior performance. An algorithm in MATLAB using the Monte Carlo simulation technique is designed to obtain numerical results. Furthermore, based on numerical results, performance evaluation measures such as average run length, extra quadratic loss, relative average run length, and comparison index are calculated. The proposed $\mathrm{ACUSUM_C}$ control charts based on performance evaluation measures and visual presentation are compared against other control charts. Findings reveal the superiority of the proposed $\mathrm{ACUSUM_C}$ control charts. Besides, for practical point of view, the proposed $\mathrm{ACUSUM_C^{(1)}}$ control chart is implemented with numerical data to show the significance over other control charts.

Keywords Adaptive · Generalized likelihood ratio test · Monte Carlo simulation · Performance evaluation measures · Score functions

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Abbreviations

ACUSUM	Adaptive CUSUM
$\mathrm{ACUSUM_c}$	Proposed control charts
$\mathrm{ACUSUM_c^{(1)}}$	$\mathrm{ACUSUM_c}$ Based on $\emptyset_h(.)$
$\mathrm{ACUSUM_c^{(2)}}$	$\mathrm{ACUSUM_c}$ Based on $\emptyset_b(.)$
$\mathrm{ACUSUM_E}$	ACUSUM based on classical EWMA statistic
AEWMA	Adaptive EWMA
$\mathrm{AEWMA_E}$	AEWMA based on classical EWMA statistic
ARL	Average run length
bmk	Benchmark
CUSUM	Cumulative sum
EQL	Extra quadratic loss
EWMA	Exponentially weighted moving average

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LCL	Lower control limit
LR	Log-likelihood ratio
OCUSUM	Optimal CUSUM
RARL	Relative average run length
RL	Run length
PCI	Performance comparison index
UCL	Upper control limit

List of Symbols

$\text{ACUSUM}_i^{(1)+}/\text{ACUSUM}_i^{(1)-}$	Plotting statistic of $\text{ACUSUM}_\text{c}^{(1)}$
$\text{ACUSUM}_i^{(2)+}/\text{ACUSUM}_i^{(2)-}$	Plotting statistic of $\text{ACUSUM}_\text{c}^{(2)}$
$\text{ARL}(\delta)$	ARL at specific shift
$\text{ARL}_\text{bmk}(\delta)$	ARL of benchmark control chart at specific shift
C_i^+/C_i^-	Plotting statistics of CUSUM
C_{i-1}^+/C_{i-1}^-	One lag values of C_i^+/C_i^-
e_i	Error
h	Control limit coefficient of CUSUM
H	Control limit of CUSUM
$h\left(\hat{\delta}_i^+\right)$	Function of $\hat{\delta}_i^+$
$h\left(\hat{\delta}_i^-\right)$	Function of $\hat{\delta}_i^-$
h_ACUSUM	Control limit coefficient of ACUSUM
H_ACUSUM	Control limit of ACUSUM
$h_{\text{ACUSUM}_\text{E}}$	Control limit coefficient of ACUSUM_E
$H_{\text{ACUSUM}_\text{E}}$	Control limit of ACUSUM_E
$h_{\text{ACUSUM}_\text{c}^{(1)}}$	Control limit coefficient of $\text{ACUSUM}_\text{c}^{(1)}$
$\pm H_{\text{ACUSUM}_\text{c}^{(1)}}$	Control limits of $\text{ACUSUM}_\text{c}^{(1)}$
$h_{\text{ACUSUM}_\text{c}^{(2)}}$	Control limit coefficient of $\text{ACUSUM}_\text{c}^{(2)}$
$\pm H_{\text{ACUSUM}_\text{c}^{(2)}}$	Control limits of $\text{ACUSUM}_\text{c}^{(2)}$
ith	Order of observation
K	Constant depends on k and σ_0
k	Constant
$N\left(\mu_0, \sigma_0^2\right)$	Normal distribution
μ	Represents μ_0 and μ_1
μ_0	Location parameter
RARL_c	RARL of a particular control chart
γ	Constant
S_i^+/S_i^-	Plotting statistics of ACUSUM
$\hat{\delta}_i$	Time-varying statistic
$\hat{\delta}_{i-1}$	One lag value of $\hat{\delta}_i$
δ_min^+	Constant and $\delta_\text{min}^+ > 0$
$\hat{\delta}_i^+$	Max function of δ_min^+ and $\hat{\delta}_i$
δ_min^-	Constant and $\delta_\text{min}^- > 0$
$\hat{\delta}_i^-$	Function of δ_min^- and $\hat{\delta}_i$
$\hat{\delta}_i^{(1)+}/\hat{\delta}_i^{(1)-}$	Function of $\delta_\text{min}^{(1)+}/\delta_\text{min}^{(1)-}$ and $w_1(e_i)$
$\delta_\text{min}^{(1)+}/\delta_\text{min}^{(1)-}$	Constant and $\delta_\text{min}^{(1)+} > 0/\delta_\text{min}^{(1)-} < 0$
$\hat{\delta}_i^{(2)+}/\hat{\delta}_i^{(2)-}$	Function of $\delta_\text{min}^{(2)+}/\delta_\text{min}^{(2)-}$ and $w_2(e_i)$
$\delta_\text{min}^{(2)+}/\delta_\text{min}^{(2)-}$	Constant and $\delta_\text{min}^{(2)+} > 0/\delta_\text{min}^{(2)-} < 0$
δ_max	Maximum shift
δ_min	Minimum shift
$w\left(\hat{\delta}_i^+\right)/w\left(\hat{\delta}_i^-\right)$	Function of $\hat{\delta}_i^+/\hat{\delta}_i^-$
$w_1(e_i)$	Ratio of $\emptyset_h(e_i)$ and e_i
$w_2(e_i)$	Ratio of Bi-square function and e_i
x_i	A process characteristic
Z_i^+/Z_i^-	Plotting statistics of ACUSUM_E
σ_0^2	Variance parameter
λ	Constant
$\emptyset_\text{h}()$	Huber function
$\emptyset_\text{b}(e_i)$	Bi-square function

1 Introduction

Manufacturing and non-manufacturing process parameters (location and/or dispersion) face two sources of variation. These sources of variation are normally categorized as random causes of variation and special causes of variation. Furthermore, random causes of variation are inherited part of every process and have a harmless nature for any process. A process is called statistically in-control under random causes of variation. In contrary, special causes of variation occur due to some problems such as problem in raw material, improper adjustment in machines, and human error. A process is called statistically out-of-control with special causes of variation. Statistically a special cause of variation is also known as a shift in the process parameters. Timely action to detect or eliminate a shift in the process parameters can bring it back into normal position.

To detect or eliminate a shift in the process parameters, the statistical process control (SPC) tools are famous. More specifically control charts among SPC tools got special attention because of their effectiveness and ease implementation. Normally control charts are categorized as memory-less and memory control charts. Shewhart control chart suggested by Shewhart [1] known as memory-less is used to monitor a shift of large size in the process parameters, but less sensitive to detect small-to-moderate sizes shift. To overcome the problem of Shewhart control chart, Page [2] and Roberts [3] offered the classical cumulative sum (CUSUM) and exponentially weighted moving average (EWMA) control charts to identify small-to-moderate shift, respectively; these control charts also known as memory control charts. The classical CUSUM control chart is only effective for a shift for which it is designed [4] by considering a specific value of its reference parameter (e.g., K). Besides, it is hard to predict the actual magnitude of the shift in real life. So, the CUSUM control chart may not perform well when future shift infrequently known or changed over the time.

To solve the issue of the classical CUSUM control chart [2], multiple classical CUSUM control charts at different values of reference parameter that can be designed to detect different sizes of shift are recommended, but multiple classical CUSUM control charts for the said purpose are difficult to design and create complications in the implementation procedures [5]. On the other hand, the classical CUSUM control chart advanced forms such as an adaptive CUSUM, denoted as ACUSUM control chart, became popular among the research community [6] to distinguish different sizes of unknown shift. There are few advantages of ACUSUM control chart over the classical CUSUM control chart. Firstly, ACUSUM control charts are easy to implement because they are comprised a single control charts as compared to multiple control charts [7]. Secondly, in the ACUSUM control charts, reference parameters are adjusted dynamically to gain better performance for a broad range of shift [8].

In this regard, an ACUSUM control chart recommended by Sparks [9] to monitor a shift in the process location. The ACUSUM control chart [9] methodology is based on the classical EWMA statistic, and it has good detection ability in a range of unknown shifts. Likewise, Jiang et al. [7] recommended an ACUSUM control chart based on classical EWMA statistic, symbolized as an ACUSUM_{E} control chart. The reference parameter of the ACUSUM_{E} control chart is first adaptively updated on the classical EWMA statistic and then a weight is assigned to it using Huber function. Recently, Abbasi and Haq [10] have extended ACUSUM_{E} [7] control chart based on auxiliary information to identify different sizes of a shift in the process location effectively. More insights into adaptive memory control charts can be seen in the studies of Capizzi and Masarotto [11], Shu and Jiang [12], Wu et al. [8], Amiri et al. [13], Zaman et al. [14], Abbasi and Haq [15], Zaman et al. [16], and references therein.

As mentioned before, an ACUSUM_{E} [7] control chart uses the classical EWMA statistic. This main issue with the classical EWMA statistic is: it does not provide clear instruction for the parameter (i.e., λ) value selections to identify a specific shift [4]. Additionally, it may be interesting and vital for researchers, quality engineers, practitioners, and experts to use the classical CUSUM statistic for the ACUSUM_{E} control chart (symbolized as an ACUSUM_{c}) against the classical EWMA statistic. May it help to further improve the detection ability for a specific as well as for a broad range of shift. So, this point is taken as an inspiration to design this study. And this study proposes two ACUSUM_{c} ($\text{ACUSUM}_{\text{c}}^{(1)}$ and $\text{ACUSUM}_{\text{c}}^{(2)}$) control charts to identify a particular also a certain range of shift. The rationality behind the proposed $\text{ACUSUM}_{\text{c}}^{(1)}and\text{ACUSUM}_{\text{c}}^{(1)}$ control charts is to first adaptively update the reference parameter based on the classical CUSUM statistic and then assign it as a weight on observation (s) using weighting functions such as Huber and Bi-square, respectively.

The performance evaluation measures such as average run length (ARL) for a specific shift, extra quadratic loss (EQL), relative average run length (RARL), and performance comparison index (PCI) to assess the overall performance of control charts are used. These performance evaluation measures are computed using simulated data through Monte Carlo simulation technique by designing an algorithm in MATLAB. The proposed $\text{ACUSUM}_{\text{c}}^{(1)}and\text{ACUSUM}_{\text{c}}^{(2)}$ control charts performance is investigated against the classical CUSUM [2], an ACUSUM [9], an adaptive EWMA (AEWMA) suggested by Capizzi and Masarotto [11], denoted as AEWMA_{E}, and ACUSUM_{E} [7] control charts. The findings reveal that the proposed $\text{ACUSUM}_{\text{C}}^{(1)}$ and $\text{ACUSUM}_{\text{C}}^{(2)}$ control charts, perform quite effective against counterparts for a specific shift as well as for a certain range of shift. Finally, the proposed control chart is also implemented on numerical data which are taken from Montgomery [17] and analysis against other control charts to show the significance.

The rest of the article is organized as follows: Sect. 2 represents the research methodologies of the classical CUSUM, ACUSUM, and ACUSUM_{E} control charts. Likewise, the proposed $\text{ACUSUM}_{\text{c}}^{(1)}$ and $\text{ACUSUM}_{\text{c}}^{(2)}$ control charts methodologies are introduced in Sect. 3. The performance evaluation measures, parameters effect on the properties of run length (RL), Monte Carlo simulation technique, and construction procedures of the proposed control charts are presented in Sect. 4. Illustration of results and comparison of the control charts based on performance evaluation measures are explained in Sect. 5. Implementation procedure of control chart with numerical data is provided in Sect. 6. Finally, summary, conclusions, and recommendations are part of Sect. 7.

2 Existing Methods

This section contains the explanation of the variable of interest and generalization Markovian of classical EWMA statistic. Besides, the basic structures of some existing control charts such as the classical CUSUM, ACUSUM, and ACUSUM_{E} control charts to monitor the process location shift are also defined.

2.1 Variable of Interest

Let X $(x_i \sim N(\mu_0, \sigma_0^2),\ i = 1, 2, 3 \ldots n)$ denote a process characteristic of interest that follows a normal probability distribution with known in-control location μ_0 (location parameter) and variance σ_0^2.

2.2 Classical CUSUM Control Chart

Page [2] designed the classical CUSUM control chart that is famous to diagnose small-to-moderate size shift in the process location. The design structure of the classical CUSUM control chart for monitoring the i^{th} observation can be presented as follows:

$$C_i^+ = \max[0,\ C_{i-1}^+ + x_i - (\mu_0 + K)], \tag{1}$$

$$C_i^- = \min[0,\ C_{i-1}^- + x_i - \mu_0 + K], \tag{2}$$

where $C_0^{\pm} = 0$ are the initial values of the C_i^+ and C_i^- statistics. The K is considered as a reference parameter, and it is defined as one-half of the magnitude of the shift (δ). It can be expressed as: $K = k\sigma_0$, where $k \in [0.25, 1.5]$ is a constant [17]. The C_i^+ and C_i^- are one-sided upper and one-sided lower plotting statistics of the classical CUSUM control chart. The upper control limit (UCL) and lower control limit (LCL) of the classical CUSUM control chart are defined as follows:

$$H = \pm h\sigma_0, \tag{3}$$

where h is control limit coefficient that depends on the value of k and pre-fixed value of ARL_0 [17]. The ARL_0 presents in-control value of ARL. Generally, the ARL has been categorized as statistically in-control (i.e., $H_0 : \mu = \mu_0$), symbolized as ARL_0 and statistically out-of-control (i.e., $H_0 : \mu = \mu_1$), denoted as ARL_1. The μ_1 represents the deviated value of μ_0. If $C_i^+ >$ H or $C_i^- < -$H, the process is considered out-of-control; otherwise, in-control.

2.3 Generalization Markovian of Classical EWMA Statistic

Sparks [9] suggested the following the classical EWMA statistic to estimate an unknown shift in advance to monitor the process location.

$$\hat{\delta}_i = (1 - \lambda)\hat{\delta}_{i-1} + \lambda x_i, \tag{4}$$

where $\hat{\delta}_i$ is a time-varying statistic and $\hat{\delta}_0 = 0$, and the $\lambda \in (0, 1]$ is a constant. Further, to enhance the estimation for a large shift, Yashchin [18] recommended generalization Markovian of the EWMA statistic which is defined as follows:

$$\hat{\delta}_i = \hat{\delta}_{i-1} + \emptyset_h(e_i), \tag{5}$$

where $e_i = x_i - \hat{\delta}_{i-1}$ is known as an error and $\emptyset_h(e_i)$ is Huber function which is given as:

$$\emptyset_h(e_i) = \begin{cases} e_i + (1-\lambda)\gamma, & \text{if } e_i < -\gamma \\ \lambda e_i, & \text{if } |e_i| \le \gamma \\ e_i - (1-\lambda)\gamma, & \text{if } e_i > \gamma, \end{cases} \tag{6}$$

where $\gamma \in (1, 4]$ is a constant. If $\emptyset_h(e_i) = \lambda e_i$, the generalization Markovian Eq. (5) reduces to the classical EWMA statistic Eq. (4).

2.4 ACUSUM Control Chart

Sparks [9] recommended an ACUSUM control chart to diagnose a certain range of shift in the process location. The upper-sided plotting statistic of an ACUSUM control chart is defined as:

$$S_i^+ = \max\left[0,\ S_{i-1}^+ + \left(x_i - \hat{\delta}_i^+/2\right)\Big/h\left(\hat{\delta}_i^+\right)\right], \tag{7}$$

where $\hat{\delta}_i^+ = max\left(\delta_{\min}^+,\ \hat{\delta}_i\right)$, $\delta_{\min}^+ > 0$, and $h\left(\hat{\delta}_i^+\right)$ is a function that defines the control limit for the high-side statistic given by Eq. (1). Similarly, the lower-sided plotting statistic of an ACUSUM control chart is designed as follows:

$$S_i^- = \max\left[0,\ S_{i-1}^- + \left(x_i - \hat{\delta}_i^-/2\right)\Big/h\left(-\hat{\delta}_i^-\right)\right], \tag{8}$$

where $\hat{\delta}_i^- = \min\left(\delta_{\min}^-,\ \hat{\delta}_i\right)$ and $\delta_{\min}^- < 0$. The $h\left(\hat{\delta}_i^-\right)$ has the same interpretation as $h\left(\hat{\delta}_i^+\right)$ has. More details on $h\left(\hat{\delta}_i^{\pm}\right)$ can be found in the study of Sparks [9]. The UCL and LCL of the ACUSUM control chart are given below:

$$H_{\text{ACUSUM}} = \pm h_{\text{ACUSUM}}\sigma_0, \tag{9}$$

where h_{ACUSUM} is control limit coefficient depends on pre-defined value ARL_0. If $S_i^+ > H_{\text{ACUSUM}}$ or $S_i^- < -H_{\text{ACUSUM}}$, the process is out-of control; otherwise, in-control.

2.5 ACUSUM_E Control Chart

The ACUSUM_E control chart proposed by Jiang et al. [7] is effective to detect a certain range of shifts in the process location. The upper-sided plotting statistic design structure of the ACUSUM_E control chart is defined as follows:

$$Z_i^+ = \max\left[0,\ Z_{i-1}^+ + w\left(\hat{\delta}_i^+\right)\left(x_i - \hat{\delta}_i^+/2\right)\right], \tag{10}$$

where $\hat{\delta}_i^+ = \max\left(\delta_{\min}^+,\ \hat{\delta}_i\right)$, $\delta_{\min}^+ > 0$, and $w\left(\hat{\delta}_i^+\right) = \hat{\delta}_i^+$ is a linear weight function. Analogously, the lower plotting statistic design structure of the ACUSUM_E control chart can be presented as follows:

$$Z_i^- = \min\left[0,\ Z_{i-1}^- - w\left(\hat{\delta}_i^-\right)\left(x_i - \hat{\delta}_i^-/2\right)\right], \quad (11)$$

where $\hat{\delta}_i^- = \min\left(\delta_{\min}^-, \hat{\delta}_i\right)$, $\delta_{\min}^- < 0$, and $w\left(\hat{\delta}_i^-\right) = \hat{\delta}_i^-$. The control limits (UCL and LCL) of the ACUSUM_E control chart are defined as follows:

$$H_{\text{ACUSUM}_\text{E}} = \pm h_{\text{ACUSUM}_\text{E}} \sigma_0 \quad (12)$$

where $h_{\text{ACUSUM}_\text{E}}$ is known as control limit coefficient that depends on pre-defined value ARL_0. If $Z_i^+ > H_{\text{ACUSUM}_\text{E}}$ or $Z_i^- < -H_{\text{ACUSUM}_\text{E}}$, the process is out-of control; otherwise, in-control.

2.6 ACUSUM as a Special Case of ACUSUM_E Control Chart

Comparing the $h\left(\hat{\delta}_i^{\pm}\right)$ of ACUSUM and $w\left(\hat{\delta}_i^{\pm}\right)$ of ACUSUM_E as a function of $\hat{\delta}_i$, it can be observed that the $h\left(\hat{\delta}_i^{\pm}\right)$ is the curvilinear function, while $w\left(\hat{\delta}_i^{\pm}\right)$ is a linear function [7]. Therefore, it corresponds to the ACUSUM_E control chart statistics in Eq. (3) using weight function $w\left(\hat{\delta}_i^+\right) = 1/h\left(\hat{\delta}_i^+\right)$.

3 Proposed ACUSUM_c Control Charts

This section comprises the methodologies of the proposed $\text{ACUSUM}_\text{c}^{(1)}$ and $\text{ACUSUM}_\text{c}^{(2)}$ control charts. In more details, the methodologies of the proposed $\text{ACUSUM}_\text{c}^{(1)}$ and $\text{ACUSUM}_\text{c}^{(2)}$ control charts are based on the classical CUSUM statistic and Huber and Bi-square, respectively. The conceptual framework is to adjust the reference parameters dynamically by allocating weights on them for a broad detection of shifts in the process location. Furthermore, it is theoretically attractive because it is loosely built on the generalized likelihood ratio test (see "Appendix"). More details on the designing of the proposed $\text{ACUSUM}_\text{c}^{(1)}$ and $\text{ACUSUM}_\text{c}^{(2)}$ control charts are provided in the following subsections.

3.1 Proposed $\text{ACUSUM}_\text{c}^{(1)}$ Control Chart

The proposed $\text{ACUSUM}_\text{c}^{(1)}$ control chart design structure is based on the classical CUSUM statistic and Huber function. The upper one-sided plotting statistic of the proposed $\text{ACUSUM}_\text{c}^{(1)}$ control chart can be defined as follows:

$$\text{ACUSUM}_i^{(1)+} = \max\left[0,\ \text{ACUSUM}_{i-1}^{(1)+} + \hat{\delta}_i^{(1)+}\left(x_i - \hat{\delta}_i^{(1)+}/2\right)\right], \quad (13)$$

where $\hat{\delta}_i^{(1)+} = \max\left(\delta_{\min}^{(1)+}, w_1(e_i)\right)$ is time-varying weight, the $\text{ACUSUM}_{i-1}^{(1)+}$ is one lag value of $\text{ACUSUM}_i^{(1)+}$, the $\delta_{\min}^{(1)+}$ is a specific value and should be positive (i.e., $\delta_{\min}^{(1)+} > 0$), and $w_1(e_i) = \emptyset_h(e_i)/e_i$ is the ratio of $\emptyset_h(e_i)$ and e_i (e.g., $e_i = x_i - C_{i-1}^+$) error. Similarly, the analogously lower one-sided plotting statistic of the proposed $\text{ACUSUM}_\text{c}^{(1)}$ control chart can be presented as follows:

$$\text{ACUSUM}_i^{(1)-} = \min\left[0,\ \text{ACUSUM}_{i-1}^{(1)-} - \hat{\delta}_i^{(1)-}\left(x_i - \hat{\delta}_i^{(1)-}/2\right)\right], \quad (14)$$

where $\hat{\delta}_i^{(1)-} = \min\left(\delta_{\min}^{(1)-}, -|w_1(e_i)|\right)$ is also time-varying weight and the $\delta_{\min}^{(1)-}$ is a specific value and should be negative (i.e., $\delta_{\min}^{(1)-} < 0$) and $e_i = x_i - C_{i-1}^-$ is error. The control limits of the proposed $\text{ACUSUM}_\text{c}^{(1)}$ control chart are designed as below:

$$H_{\text{ACUSUM}_\text{c}^{(1)}} = \pm h_{\text{ACUSUM}_\text{c}^{(1)}} \sigma_0, \quad (15)$$

where $h_{\text{ACUSUM}_\text{c}^{(1)}}$ is a control limit coefficient (see Table 1) which depends on pre-defined value of ARL_0. If $\text{ACUSUM}_i^{(1)+} > H_{\text{ACUSUM}_\text{c}^{(1)}}$ or $\text{ACUSUM}_i^{(1)-} < -H_{\text{ACUSUM}_\text{c}^{(1)}}$, the process is out-of-control; otherwise, in-control.

3.2 Proposed $\text{ACUSUM}_\text{c}^{(2)}$ control chart

The proposed $\text{ACUSUM}_\text{c}^{(2)}$ control chart methodology is based on the classical CUSUM statistic and Bi-square function. The upper one-sided plotting statistic of the proposed $\text{ACUSUM}_\text{c}^{(2)}$ control chart is given as follows:

$$\text{ACUSUM}_i^{(2)+} = \max\left[0,\ \text{ACUSUM}_i^{(2)+} + \hat{\delta}_i^{(2)+}\left(x_i - \hat{\delta}_i^{(2)+}/2\right)\right], \quad (16)$$

where $\hat{\delta}_i^{(2)+} = \max\left(\delta_{\min}^{(2)+}, w_2(e_i)\right)$ is time-varying weight and the $\delta_{\min}^{(2)+}$ is a specific value and should be positive (i.e., $\delta_{\min}^{(2)+} > 0$), and $w_2(e_i) = \emptyset_b(e_i)/e_i$ is the ratio of $\emptyset_b(e_i)$ (Bi-square function) and error e_i. The Bi-square function can be

Table 1 Control limits coefficient $h_{\text{ACUSUM}_c^{(1)}}$ values at $\delta_{\min}^{+} = 0.50$ when $\text{ARL}_0 = 400$

k	λ	γ	$h_{\text{ACUSUM}_c^{(1)}}$	γ	$h_{\text{ACUSUM}_c^{(1)}}$	γ	$h_{\text{ACUSUM}_c^{(1)}}$	γ	$h_{\text{ACUSUM}_c^{(1)}}$	γ	$h_{\text{ACUSUM}_c^{(1)}}$	γ	$h_{\text{ACUSUM}_c^{(1)}}$
0.10	0.10	1.00	3.33	1.50	3.19	2.00	3.18	2.50	3.16	3.00	3.20	4.00	3.30
0.20	0.10	1.00	3.11	1.50	3.04	2.00	3.05	2.50	3.17	3.00	3.26	4.00	3.40
0.25	0.10	1.00	3.06	1.50	3.02	2.00	3.05	2.50	3.21	3.00	3.30	4.00	3.43
0.30	0.10	1.00	3.06	1.50	3.03	2.00	3.10	2.50	3.26	3.00	3.38	4.00	3.42
0.40	0.10	1.00	3.05	1.50	3.12	2.00	3.20	2.50	3.38	3.00	3.42	4.00	3.43
0.50	0.10	1.00	3.14	1.50	3.23	2.00	3.30	2.50	3.40	3.00	3.40	4.00	3.43
0.10	0.20	1.00	3.35	1.50	3.23	2.00	3.18	2.50	3.15	3.00	3.20	4.00	3.25
0.20	0.20	1.00	3.14	1.50	3.07	2.00	3.05	2.50	3.12	3.00	3.18	4.00	3.34
0.25	0.20	1.00	3.12	1.50	3.00	2.00	3.05	2.50	3.15	3.00	3.24	4.00	3.40
0.30	0.20	1.00	3.08	1.50	3.00	2.00	3.10	2.50	3.20	3.00	3.31	4.00	3.42
0.40	0.20	1.00	3.08	1.50	3.08	2.00	3.20	2.50	3.32	3.00	3.38	4.00	3.43
0.50	0.20	1.00	3.14	1.50	3.18	2.00	3.30	2.50	3.38	3.00	3.41	4.00	3.43
0.10	0.30	1.00	3.40	1.50	3.27	2.00	3.18	2.50	3.16	3.00	3.15	4.00	3.20
0.20	0.30	1.00	3.23	1.50	3.07	2.00	3.05	2.50	3.07	3.00	3.13	4.00	3.27
0.25	0.30	1.00	3.16	1.50	3.02	2.00	3.02	2.50	3.09	3.00	3.17	4.00	3.32
0.30	0.30	1.00	3.12	1.50	3.00	2.00	3.05	2.50	3.13	3.00	3.22	4.00	3.40
0.40	0.30	1.00	3.10	1.50	3.04	2.00	3.13	2.50	3.23	3.00	3.34	4.00	3.42
0.50	0.30	1.00	3.14	1.50	3.16	2.00	3.30	2.50	3.33	3.00	3.38	4.00	3.43
0.10	0.40	1.00	3.46	1.50	3.31	2.00	3.23	2.50	3.19	3.00	3.15	4.00	3.18
0.20	0.40	1.00	3.28	1.50	3.11	2.00	3.04	2.50	3.05	3.00	3.09	4.00	3.20
0.25	0.40	1.00	3.23	1.50	3.07	2.00	3.00	2.50	3.03	3.00	3.09	4.00	3.25
0.30	0.40	1.00	3.19	1.50	3.03	2.00	3.01	2.50	3.06	3.00	3.14	4.00	3.31
0.40	0.40	1.00	3.16	1.50	3.04	2.00	3.07	2.50	3.16	3.00	3.26	4.00	3.38
0.50	0.40	1.00	3.22	1.50	3.14	2.00	3.19	2.50	3.28	3.00	3.36	4.00	3.42
0.10	0.50	1.00	3.52	1.50	3.39	2.00	3.28	2.50	3.22	3.00	3.19	4.00	3.15
0.20	0.50	1.00	3.36	1.50	3.17	2.00	3.07	2.50	3.05	3.00	3.06	4.00	3.12
0.25	0.50	1.00	3.30	1.50	3.12	2.00	3.04	2.50	3.03	3.00	3.03	4.00	3.14
0.30	0.50	1.00	3.27	1.50	3.10	2.00	3.02	2.50	3.02	3.00	3.06	4.00	3.20
0.40	0.50	1.00	3.25	1.50	3.08	2.00	3.00	2.50	3.09	3.00	3.16	4.00	3.32
0.50	0.50	1.00	3.20	1.50	3.16	2.00	3.15	2.50	3.18	3.00	3.36	4.00	3.38

defined as:

$$\emptyset_b(e_i) = \begin{cases} e_i\left(1-(1-\lambda)\left[1-(e_i/\gamma)^2\right]^2\right) & \text{if } |e_i| \le \gamma \\ e_i, & \text{otherwise} \end{cases}. \tag{17}$$

Similarly, the lower one-sided plotting statistic of the proposed $\text{ACUSUM}_c^{(2)}$ control chart is given as:

$$\text{ACUSUM}_i^{(2)-} = \min\left[[0,\ \text{ACUSUM}_i^{(2)-} - \hat{\delta}_i^{(2)-}\left(x_i - \hat{\delta}_i^{(2)-}/2\right)\right], \tag{18}$$

where $\hat{\delta}_i^{(2)-} = \min\left(\delta_{\min}^{(2)-}, -|w_2(e_i)|\right)$ is time-varying weight and the $\delta_{\min}^{(2)-}$ is a specific value and should be negative (i.e., $\delta_{\min}^{(2)-} < 0$), and $e_i = x_i - C_{i-1}^{-}$. The control limits of the proposed $\text{ACUSUM}_c^{(2)}$ control chart are designed as follows:

$$H_{\text{ACUSUM}_c^{(2)}} = \pm h_{\text{ACUSUM}_c^{(2)}}\sigma_0, \tag{19}$$

where $h_{\text{ACUSUM}_c^{(2)}}$ is known as control limit coefficient (see Table 2) that depends on pre-defined value ARL_0. If $\text{ACUSUM}_i^{(2)+} > H_{\text{ACUSUM}_c^{(2)}}$ or $\text{ACUSUM}_i^{(2)+} < -H_{\text{ACUSUM}_c^{(2)}}$, the process is out-of-control; otherwise, in-control.

Table 2 Control limits coefficient $h_{\text{ACUSUM}_c^{(2)}}$ values at $\delta^+_{\min} = 0.50$ when $\text{ARL}_0 = 400$

k	λ	γ	$h_{\text{ACUSUM}_c^{(2)}}$	γ	$h_{\text{ACUSUM}_c^{(2)}}$	γ	$h_{\text{ACUSUM}_c^{(2)}}$	γ	$h_{\text{ACUSUM}_c^{(2)}}$	γ	$h_{\text{ACUSUM}_c^{(2)}}$	γ	$h_{\text{ACUSUM}_c^{(2)}}$
0.10	0.10	1.00	4.16	1.50	4.12	2.00	4.06	2.50	3.99	3.00	3.94	4.00	3.86
0.20	0.10	1.00	4.49	1.50	4.38	2.00	4.26	2.50	4.09	3.00	3.92	4.00	3.60
0.25	0.10	1.00	4.64	1.50	4.48	2.00	4.26	2.50	4.05	3.00	3.81	4.00	3.39
0.30	0.10	1.00	4.79	1.50	4.52	2.00	4.22	2.50	3.96	3.00	3.69	4.00	3.19
0.40	0.10	1.00	4.99	1.50	4.56	2.00	4.15	2.50	3.76	3.00	3.44	4.00	3.02
0.50	0.10	1.00	5.13	1.50	5.13	2.00	4.10	2.50	3.68	3.00	3.38	4.00	3.08
0.10	0.20	1.00	4.18	1.50	4.15	2.00	4.09	2.50	4.12	3.00	3.98	4.00	3.90
0.20	0.20	1.00	4.52	1.50	4.40	2.00	4.27	2.50	4.08	3.00	3.98	4.00	3.63
0.25	0.20	1.00	4.69	1.50	4.50	2.00	4.26	2.50	4.01	3.00	3.39	4.00	3.39
0.30	0.20	1.00	4.82	1.50	4.54	2.00	4.27	2.50	3.80	3.00	3.22	4.00	3.22
0.40	0.20	1.00	5.02	1.50	4.58	2.00	4.17	2.50	3.80	3.00	3.48	4.00	3.03
0.50	0.20	1.00	5.15	1.50	4.60	2.00	4.13	2.50	4.09	3.00	3.39	4.00	3.10
0.10	0.30	1.00	4.23	1.50	4.19	2.00	4.14	2.50	4.19	3.00	4.02	4.00	3.92
0.20	0.30	1.00	4.54	1.50	4.44	2.00	4.33	2.50	4.08	3.00	4.02	4.00	3.66
0.25	0.30	1.00	4.73	1.50	4.55	2.00	4.36	2.50	4.05	3.00	3.92	4.00	3.45
0.30	0.30	1.00	4.83	1.50	4.59	2.00	4.33	2.50	3.83	3.00	3.76	4.00	3.26
0.40	0.30	1.00	5.03	1.50	4.62	2.00	4.23	2.50	3.74	3.00	3.51	4.00	3.08
0.50	0.30	1.00	5.13	1.50	4.64	2.00	4.15	2.50	4.15	3.00	3.45	4.00	3.10
0.10	0.40	1.00	4.25	1.50	4.24	2.00	4.18	2.50	4.28	3.00	4.10	4.00	4.02
0.20	0.40	1.00	4.61	1.50	4.50	2.00	4.43	2.50	4.22	3.00	4.13	4.00	3.76
0.25	0.40	1.00	4.75	1.50	4.62	2.00	4.36	2.50	4.12	3.00	4.00	4.00	3.49
0.30	0.40	1.00	4.89	1.50	4.65	2.00	4.40	2.50	3.92	3.00	3.86	4.00	3.32
0.40	0.40	1.00	5.05	1.50	4.65	2.00	4.25	2.50	3.74	3.00	3.59	4.00	3.15
0.50	0.40	1.00	5.115	1.50	4.64	2.00	4.20	2.50	4.25	3.00	3.49	4.00	3.17
0.10	0.50	1.00	4.30	1.50	4.30	2.00	4.29	2.50	4.40	3.00	4.22	4.00	4.15
0.20	0.50	1.00	4.65	1.50	4.59	2.00	4.49	2.50	4.22	3.00	4.25	4.00	3.88
0.25	0.50	1.00	4.82	1.50	4.70	2.00	4.53	2.50	4.25	3.00	4.13	4.00	3.68
0.30	0.50	1.00	4.95	1.50	4.73	2.00	4.50	2.50	4.01	3.00	3.99	4.00	3.46
0.40	0.50	1.00	5.10	1.50	4.76	2.00	4.37	2.50	3.74	3.00	3.69	4.00	3.26
0.50	0.50	1.00	5.16	1.50	4.69	2.00	4.25	2.50	3.87	3.00	3.59	4.00	3.30

3.3 Monte Carlo Simulations Technique

Based on Monte Carlo simulation [19] procedure, an algorithm is developed in MATLAB for the numerical results and to compute ARL values. Monte Carlo simulation procedures with 10^5 repetitions are carried out for each change in δ. The shift δ is set between 1 and 4. The $h_{\text{ACUSUM}_c^{(1)}}$ and $h_{\text{ACUSUM}_c^{(2)}}$ at $\delta^+_{\min} = 0.50$ with possible combinations of k, λ, and γ are presented in Tables 1 and 2, respectively, for a given $\text{ARL}_0 = 400$. Only upper one-sided scenario is considered because the lower one-sided scenario provides the same behavior as an upward by same absolute amount shift. Besides, the EQL, RARL, and PCI are also calculated using ARLs (see Sect. 4).

3.4 Construction Procedure of $ACUSUM_c$ Control Charts

This subsection explains the construction procedures of the proposed $\text{ACUSUM}_c^{(1)}$ and $\text{ACUSUM}_c^{(2)}$ control charts. More details are given as follows:

3.4.1 $ACUSUM_c^{(1)}$ Control Chart

Suppose that the ARL_0 is 400 for upper one-sided $\text{ACUSUM}_i^{(1)+}$ statistic. So, the procedure to construct the proposed $\text{ACUSUM}_c^{(1)}$ control chart is given in the following steps:

(i) An infinite loop is applied to generate $x_i \sim N(\mu_0, \sigma_0^2)$ with $i = 1, 2 \ldots$
(ii) Calculate the C_{i-1}^{+} statistic from Eq. (1) using x_i and K.
(iii) Compute the $e_i = x_i - C_{i-1}^{+}$ statistic and select $\emptyset_h(e_i)$ from Eq. (6) based on the relation of the e_i statistic and γ (i.e., $\gamma \in (1, 4)$) constant.
(iv) Calculate the $\hat{\delta}_i^{(1)+} = \max\left(\delta_{\min}^{(1)+}, w_1(e_i)\right)$ where $\delta_{\min}^{(1)+} > 0$ and $w_1(e_i) = \emptyset_h(e_i)/e_i$.
(v) Calculate the $ACUSUM_i^{(1)+}$ statistic from Eq. (13) based on x_i and $\hat{\delta}_i^{(1)+}$ while $ACUSUM_0^{(1)+} = 0$.
(vi) Let us assume $h_{\text{ACUSUM}_c^{(1)}} = 3$.
(vii) Calculate the $H_{\text{ACUSUM}_c^{(1)}}$ control limits from Eq. (15) based σ_0.
(viii) Plot the $ACUSUM_i^{(1)+}$ statistic against $H_{\text{ACUSUM}_C^{(1)}}$ over i.
(ix) If the $ACUSUM_i^{(1)+} > H_{\text{ACUSUM}_c^{(1)}}$, note sample number of $ACUSUM_i^{(1)+}$ statistic as a RL. For example, at $i = 355$, if $ACUSUM_{355}^{(1)+} > H_{\text{ACUSUM}_c^{(1)}}$ record 355 is as a first RL.
(x) Repeat from (i)–(ix) steps for 10^5 times and record RLs.
(xi) Compute the average of 10^5 noted RLs, which is called ARL_0.
(xii) If $\text{ARL}_0 = 400$; otherwise, adjust $h_{\text{ACUSUM}_c^{(1)}}$ (i.e.$h_{\text{ACUSUM}_c^{(1)}} < 3$ or $h_{\text{ACUSUM}_c^{(1)}} > 3$) constant accordingly in step (vi) and repeat from (i)-(xi) steps to obtain $\text{ARL}_0 = 400$.
(xiii) To compute the ARL_1 values, generate $x_i \sim N(\mu_1, \sigma_0^2)$ $(\delta = \mu_1 > \mu_0)$ with $i = 1, 2 \ldots$ and repeat from (ii)-(xi) steps.

Similarly, the lower one-sided and two-sided proposed $\text{ACUSUM}_c^{(1)}$ control charts can be constructed with similar lines using respective statistics, parameters, and constants.

3.4.2 *ACUSUM*$_c^{(2)}$ Control Chart

Like $\text{ACUSUM}_c^{(1)}$ control chart, the proposed $\text{ACUSUM}_c^{(2)}$ control chart can be constructed by following the given below guidelines.

(i) An infinite loop is applied to generate $x_i \sim N(\mu_0, \sigma_0^2)$ with $i = 1, 2 \ldots$
(ii) Calculate the C_{i-1}^{+} statistic from Eq. (1) using the x_i statistic and K constant.
(iii) Compute the $e_i = x_i - C_{i-1}^{+}$ statistic and select $\emptyset_b(e_i)$ from Eq. (17) based on the relation of the e_i statistic and γ ($\gamma \in (1, 4)$) constant.
(iv) Calculate the $\hat{\delta}_i^{(2)+} = \max\left(\delta_{\min}^{(2)+}, w_b(e_i)\right)$ where $\delta_{\min}^{(2)+} > 0$ and $w_2(e_i) = \emptyset_b(e_i)/e_i$.
(v) Calculate the $ACUSUM_i^{(2)+}$ statistic from Eq. (16) based on x_i statistic and $\hat{\delta}_i^{(2)+}$ while $ACUSUM_0^{(2)+} = 0$.
(vi) Let us assume that $h_{\text{ACUSUM}_c^{(2)}} = 3$.
(vii) Calculate the $H_{\text{ACUSUM}_c^{(2)}}$ control limit from Eq. (19) based σ_0.
(viii) Plot the $ACUSUM_i^{(1)+}$ statistic against $H_{\text{ACUSUM}_c^{(2)}}$ over i.
(ix) If the $ACUSUM_i^{(2)+} > H_{\text{ACUSUM}_c^{(2)}}$, note the sample number of the $ACUSUM_i^{(2)}$ statistic as a RL. For example, at $i = 455$, if $ACUSUM_{455}^{(2)+} > H_{\text{ACUSUM}_c^{(2)}}$ record 455 is as a first RL.
(x) Repeat from (i)–(ix) steps for 10^5 times and record RLs.
(xi) Compute the average of 10^5 noted RLs, which is called ARL_0.
(xii) If $\text{ARL}_0 = 400$; otherwise, adjust $h_{\text{ACUSUM}_c^{(2)}}$ (i.e.$h_{\text{ACUSUM}_c^{(2)}} < 3$ or $h_{\text{ACUSUM}_c^{(2)}} > 3$) constant accordingly in step (vi) and repeat from (i)-(xi) steps to obtain $\text{ARL}_0 = 400$.
(xiii) To compute the ARL_1 values, generate $x_i \sim N(\mu_1, \sigma_0^2)$ $(\delta = \mu_1 > \mu_0,)$ with $i = 1, 2 \ldots$ and repeat from (ii)-(xi) steps.

Similarly, the lower one-sided and two-sided proposed $\text{ACUSUM}_c^{(2)}$ control charts can be constructed with similar lines using respective statistics, parameters, and constants. Additionally, a flow chart which describes the above steps through Monte Carlo simulation to construct the proposed $\text{ACUSUM}_c^{(1)}$ control chart is given in Fig. 1. Similar lines can be used for the proposed $\text{ACUSUM}_c^{(2)}$ control chart.

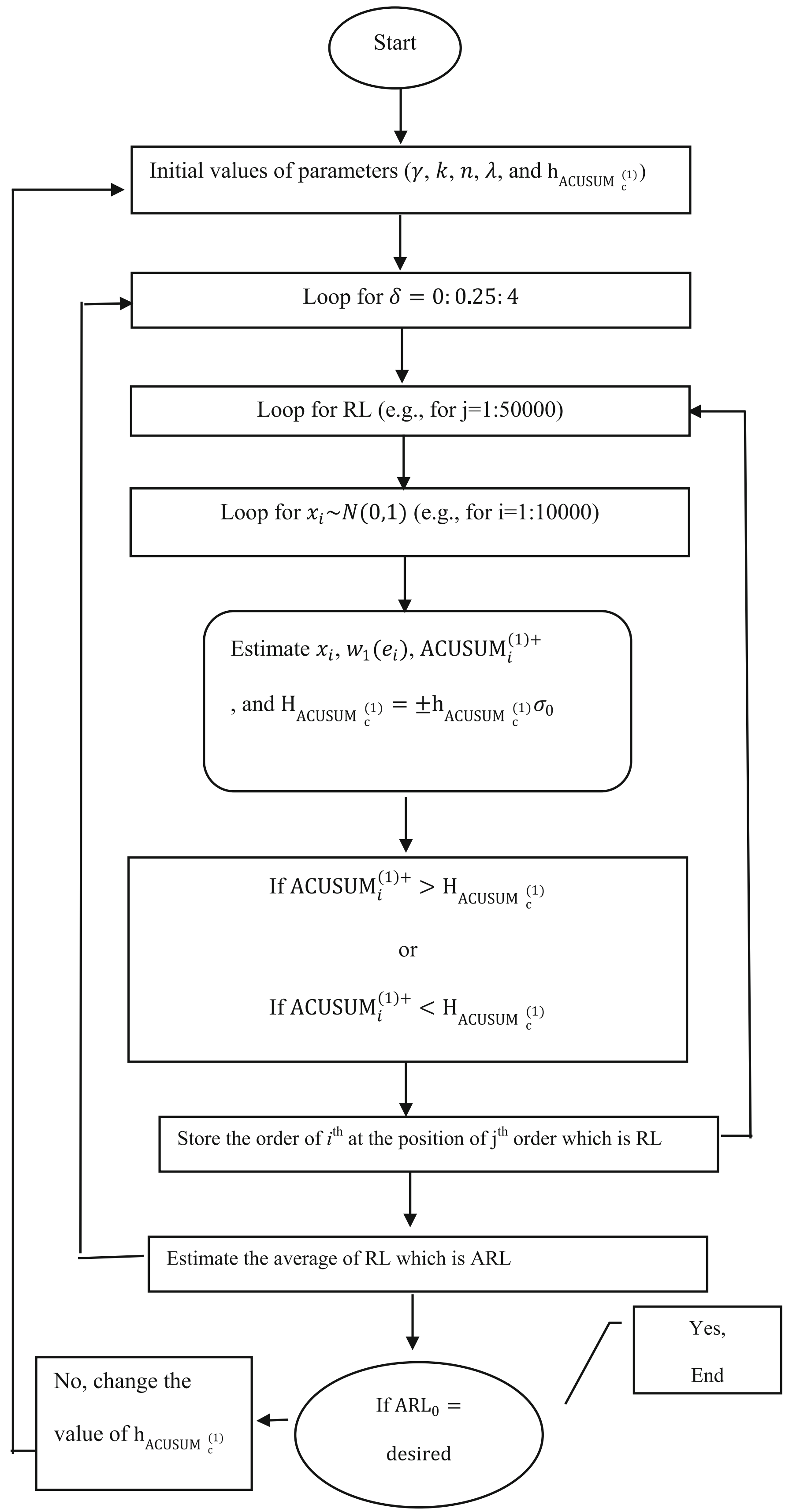
4 Performance Analysis

This section contains design structures of performance evaluation measures in Sect. 4.1. Likewise, Sect. 4.2 describes the optimal choices of the parameter's method.

4.1 Performance Evaluation Measures

This subsection illustrates different performance evaluation measures methodologies along their interpretation. The most commonly used such as ARL, EQL, RARL, and PCI measures are employed [20]. Their more details are offered in subsequent subsections.

Fig. 1 Flow chart of Monte Carlo simulation



4.1.1 Average Run Length Measure (ARL)

The ARL is a most generally used to judge a control chart assessment at a specific shift against other control charts. Usually, it has been categorized as statistically in-control (i.e., $H_0 : \mu = \mu_0$), symbolized as ARL_0 and statistically out-of-control (i.e., $H_0 : \mu = \mu_1$), denoted as ARL_1. The μ_1 represents the deviated value of μ_0. The ARL_0 value of a control chart should be at least same or greater than other control charts ARL_0 value. If the ARL_1 value is smaller at a specific δ as compared to other control chart, that control chart is considered superior.

4.1.2 Extra Quadratic Loss Measure (EQL)

The EQL provides overall performance of a control chart for a certain range of shifts. It is based on the loss function and mathematically can be defined as follows:

$$\text{EQL} = \frac{1}{\delta_{\max} - \delta_{\min}} \int_{\delta_{\min}}^{\delta_{\max}} \delta^2 \text{ARL}(\delta) \text{d}\delta,$$

where δ presents shift, the $\text{ARL}(\delta)$ is the average run length of control chart at a δ, and $\delta_{\min}$ and $\delta_{\max}$ are minimum and maximum values of δ, respectively. The EQL integrates the overall performance of a control chart based on ARL values for all shift's domain $\delta_{\min} < \delta < \delta_{\max}$, using the δ^2 as a weight.

4.1.3 Relative Average Run Length (RARL)

Like EQL, the RARL also presents the overall effectiveness of a control chart against others. Mathematically, the RARL is defined as follows:

$$\text{RARL} = \frac{1}{\delta_{\max} - \delta_{\min}} \int_{\delta_{\min}}^{\delta_{\max}} \frac{\text{ARL}(\delta)}{\text{ARL}_{\text{bmk}}(\delta)} \text{d}\delta,$$

where $\text{ARL}_{\text{bmk}}(\delta)$ presents ARL value of a benchmark (i.e., bmk) control chart. The control chart with minimum ARL_1 value is considered as a benchmark control chart. The RARL shows how a particular control chart performs closely relative to a benchmark control chart.

4.1.4 Performance Comparison Index (PCI)

Generally, the PCI is defined as the ratio of the RARL of a control chart and a control chart having smaller RARL ($\text{RARL}_{\text{benchmark}}$). It is given as below:

$$\text{PCI} = \frac{\text{RARL}_{\text{c}}}{\text{RARL}_{\text{bmk}}}.$$

If the PCI is equal to one, it is meant that control chart 'C' has minimum RARL, and it is considered as the most effective against other control charts.

4.2 Parameter Effect on Properties of RL

The optimum choices of k, λ, andγ parameters along $h_{\text{ACUSUM}_c^{(1)}}$ and $h_{\text{ACUSUM}_c^{(2)}}$ control limit coefficients may will serve the vital role to perform the proposed $\text{ACUSUM}_c^{(1)}$ and $\text{ACUSUM}_c^{(2)}$ control charts more efficiently. Therefore, first the sole effect of k, λ, and γ parameters on the classical memory control charts is explained to find out the optimum choices of k, λ, andγ parameters with $h_{\text{ACUSUM}_c^{(1)}}$ and $h_{\text{ACUSUM}_c^{(2)}}$. For instance, the classical CUSUM control chart k parameter helps to find out a specific shift. Likewise, the ranges of λ and γ parameters are $1 \leq \gamma \leq 4$ and $0 < \lambda \leq 1$, respectively [7, 11]. The main purpose to define the ranges of parameters is to balance the sensitivity of the shift. Even though this, it is very hard to find out the joint optimal combinations of k, λ, andγ parameters along $h_{\text{ACUSUM}_c^{(1)}}$ and $h_{\text{ACUSUM}_c^{(2)}}$ for sole control chart performance. However, some researchers such as Capizzi and Masarotto [11], Abbas et al. [21], and Zaman et al. [22] have explained the methodologies for it. So, to find out the optimal combination of parameters firstly it needs to find out the possible combinations of parameters based on their given ranges. More details are given in the following subsection. Additionally, the performance of the proposed $\text{ACUSUM}_c^{(1)}$ and $\text{ACUSUM}_c^{(2)}$ control charts also varying if there are changes in their parameter's values (see Tables 1 and 2) because the objective is to detect out-of-control signals early as compared to other control charts by utilizing different combinations of parameters values.

5 Illustration of Results and Performance Comparison

The comparison between the proposed $\text{ACUSUM}_c^{(1)}$ and $\text{ACUSUM}_c^{(2)}$ control charts gives an idea when the researchers, experts, and practitioners can use the proposed control charts and it is provided in Sect. 5.1. Likewise, performance evaluation comparison of the proposed $\text{ACUSUM}_c^{(1)}$ and $\text{ACUSUM}_c^{(2)}$ control charts against other control charts is offered in Sect. 5.2.

5.1 Comparison Between Proposed $\text{ACUSUM}_c^{(1)}$ and $\text{ACUSUM}_c^{(2)}$ Control Charts

This subsection contains the performance evaluation comparison between the proposed $\text{ACUSUM}_c^{(1)}$ and $\text{ACUSUM}_c^{(2)}$

Table 3 ARLs properties of proposed $ACUSUM_c^{(1)}$ control chart at different values of k, λ, and γ when $ARL_0 = 400$ and $\delta^+_{min} = 0.50$

k	0.10	0.20	0.25	0.30	0.40	0.50	0.10	0.20	0.25	0.30	0.40	0.50	0.10	0.20	0.25	0.30	0.40	0.50
λ	0.10	0.10	0.10	0.10	0.10	0.10	0.20	0.20	0.20	0.20	0.20	0.20	0.30	0.30	0.30	0.30	0.30	0.30
γ	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
0.00	407	399	402	408	407	402	397	397	402	407	407	400	398	407	407	402	400	407
0.25	70.8	73.3	75.8	80.7	79.4	73.1	70.8	72.9	77.4	79.6	81.1	74.0	72.0	74.1	77.7	79.7	82.6	77.9
0.50	27.0	26.3	26.3	27.0	27.1	26.1	26.8	26.0	26.5	26.9	27.3	26.2	26.7	26.5	26.7	26.8	27.6	26.6
0.75	14.4	13.8	13.9	14.0	14.0	14.0	14.3	13.8	13.8	14.0	14.0	13.9	14.3	14.0	13.8	13.9	13.9	14.0
1.00	9.48	9.09	9.06	9.04	9.11	9.25	9.36	9.06	8.99	9.01	9.08	9.18	9.30	9.12	8.98	8.93	9.02	9.15
1.25	7.04	6.81	6.71	6.72	6.70	6.91	6.95	6.67	6.63	6.68	6.68	6.84	6.90	6.74	6.65	6.57	6.60	6.75
1.50	5.65	5.37	5.34	5.35	5.35	5.53	5.56	5.31	5.23	5.29	5.29	5.37	5.51	5.36	5.26	5.23	5.21	5.35
1.75	4.74	4.48	4.44	4.45	4.45	4.57	4.64	4.43	4.38	4.38	4.42	4.47	4.61	4.45	4.37	4.33	4.32	4.41
2.00	4.07	3.85	3.81	3.82	3.79	3.90	3.99	3.80	3.73	3.74	3.76	3.80	3.93	3.79	3.74	3.72	3.69	3.77
2.25	3.54	3.36	3.33	3.31	3.31	3.40	3.49	3.32	3.25	3.26	3.27	3.32	3.44	3.30	3.26	3.22	3.20	3.27
2.50	3.16	2.99	2.96	2.94	2.93	3.01	3.10	2.94	2.88	2.89	2.89	2.94	3.07	2.92	2.88	2.84	2.84	2.88
2.75	2.83	2.67	2.64	2.63	2.62	2.69	2.79	2.63	2.57	2.58	2.57	2.61	2.75	2.62	2.57	2.54	2.51	2.57
3.00	2.56	2.40	2.37	2.38	2.36	2.42	2.51	2.37	2.32	2.33	2.32	2.36	2.48	2.37	2.31	2.29	2.27	2.32
3.25	2.34	2.17	2.15	2.16	2.14	2.21	2.28	2.15	2.11	2.11	2.10	2.15	2.25	2.15	2.10	2.08	2.06	2.11
3.50	2.13	2.00	1.96	1.96	1.96	2.01	2.10	1.96	1.92	1.93	1.93	1.95	2.06	1.97	1.92	1.90	1.89	1.92
3.75	1.96	1.83	1.80	1.81	1.80	1.84	1.92	1.81	1.76	1.77	1.77	1.80	1.90	1.80	1.77	1.74	1.74	1.76
4.00	1.80	1.69	1.66	1.66	1.65	1.70	1.76	1.66	1.63	1.63	1.63	1.66	1.75	1.66	1.63	1.61	1.59	1.64

k	0.10	0.20	0.25	0.30	0.40	0.50	0.10	0.20	0.25	0.30	0.40	0.50	0.10	0.20	0.25	0.30	0.40	0.50
λ	0.40	0.40	0.40	0.40	0.40	0.40	0.50	0.50	0.50	0.50	0.50	0.50	0.10	0.10	0.10	0.10	0.10	0.10
γ	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.50	1.50	1.50	1.50	1.50	1.50
0.00	401	406	405.0	399	399	400	397	400	399	398	402	397	404	397	395.9	401.0	406	405
0.25	71.8	74.6	78.1	80.7	84.2	77.9	72.7	75.0	78.3	80.4	84.4	81.5	68.8	72.3	74.4	76.3	70.3	66.6
0.50	26.8	26.4	26.8	27.1	28.0	26.9	26.8	26.7	27.0	27.4	28.3	27.7	26.4	25.5	26.0	26.0	25.3	24.5
0.75	14.2	14.0	13.90	13.8	14.0	14.1	14.2	14.0	13.8	13.9	14.1	14.2	14.2	13.6	13.78	13.76	13.8	14.0
1.00	9.30	9.06	8.94	8.98	8.94	9.11	9.25	8.97	8.91	8.95	8.95	9.04	9.46	9.14	9.07	9.16	9.39	9.56
1.25	6.81	6.63	6.57	6.54	6.61	6.64	6.79	6.60	6.55	6.52	6.53	6.61	7.14	6.85	6.83	6.89	7.04	7.27
1.50	5.43	5.28	5.24	5.18	5.15	5.27	5.39	5.23	5.16	5.14	5.16	5.21	5.73	5.53	5.51	5.57	5.67	5.86
1.75	4.56	4.39	4.33	4.26	4.26	4.33	4.48	4.32	4.27	4.25	4.24	4.28	4.85	4.65	4.65	4.65	4.78	4.92
2.00	3.90	3.75	3.70	3.68	3.62	3.68	3.82	3.70	3.64	3.62	3.59	3.65	4.18	4.01	4.01	4.02	4.13	4.24
2.25	3.41	3.25	3.23	3.19	3.17	3.21	3.36	3.23	3.19	3.15	3.14	3.16	3.70	3.54	3.53	3.54	3.62	3.73
2.50	3.03	2.89	2.86	2.81	2.78	2.84	2.98	2.85	2.81	2.79	2.76	2.81	3.29	3.16	3.15	3.16	3.23	3.34
2.75	2.71	2.59	2.55	2.52	2.49	2.53	2.67	2.55	2.52	2.49	2.48	2.49	2.97	2.86	2.83	2.85	2.90	3.00
3.00	2.45	2.32	2.30	2.27	2.24	2.28	2.42	2.31	2.28	2.25	2.24	2.26	2.69	2.59	2.59	2.59	2.65	2.73
3.25	2.23	2.12	2.09	2.06	2.04	2.08	2.21	2.12	2.06	2.05	2.04	2.07	2.48	2.36	2.35	2.37	2.42	2.50
3.50	2.05	1.93	1.91	1.89	1.87	1.90	2.03	1.94	1.89	1.88	1.87	1.89	2.27	2.17	2.16	2.16	2.22	2.28
3.75	1.88	1.79	1.75	1.74	1.71	1.74	1.86	1.78	1.75	1.73	1.72	1.74	2.09	2.01	1.98	1.98	2.05	2.11
4.00	1.74	1.65	1.62	1.60	1.58	1.62	1.72	1.64	1.61	1.59	1.59	1.61	1.93	1.85	1.8291	1.8403	1.88	1.95

Table 3 (continued)

k	0.10	0.20	0.25	0.30	0.40	0.50	0.10	0.20	0.25	0.30	0.40	0.50	0.10	0.20	0.25	0.30	0.40	0.50
λ	0.20	0.20	0.20	0.20	0.20	0.20	0.30	0.30	0.30	0.30	0.30	0.30	0.40	0.40	0.40	0.40	0.40	0.40
γ	1.50	1.50	1.50	1.50	1.50	1.50	1.50	1.50	1.50	1.50	1.50	1.50	1.50	1.50	1.50	1.50	1.50	1.50
0.00	406	408	396	397	406	399	404	398	398	408	401	407	397	404	403.3	399	401	398
0.25	69.4	73.3	74.6	76.6	74.0	68.5	70.6	72.8	75.8	79.1	75.1	71.1	70.6	73.5	76.1	78.5	78.3	73.4
0.50	26.0	26.1	25.9	26.2	25.7	24.8	26.5	25.8	25.9	26.5	26.1	25.5	26.5	26.0	26.5	26.6	26.9	25.8
0.75	14.2	13.8	13.6	13.6	13.9	13.9	14.3	13.9	13.7	13.7	13.8	14.0	14.2	13.7	13.79	13.8	13.8	14.0
1.00	9.43	9.15	9.04	9.04	9.28	9.54	9.45	9.18	9.04	9.07	9.09	9.39	9.42	9.07	8.99	8.99	9.05	9.26
1.25	7.08	6.90	6.77	6.80	6.95	7.15	7.09	6.77	6.74	6.71	6.79	7.03	7.02	6.73	6.73	6.66	6.70	6.91
1.50	5.71	5.50	5.41	5.44	5.59	5.74	5.69	5.42	5.37	5.43	5.42	5.63	5.59	5.38	5.37	5.29	5.33	5.52
1.75	4.80	4.66	4.56	4.55	4.68	4.79	4.77	4.56	4.50	4.51	4.55	4.72	4.70	4.48	4.45	4.43	4.44	4.57
2.00	4.15	3.99	3.92	3.93	4.03	4.12	4.10	3.92	3.86	3.88	3.89	4.01	4.05	3.85	3.82	3.78	3.78	3.89
2.25	3.64	3.52	3.45	3.44	3.53	3.61	3.60	3.44	3.37	3.40	3.41	3.51	3.55	3.37	3.32	3.28	3.31	3.41
2.50	3.25	3.12	3.07	3.07	3.14	3.22	3.20	3.03	3.00	3.01	3.02	3.12	3.14	2.99	2.95	2.91	2.92	3.00
2.75	2.93	2.82	2.75	2.76	2.82	2.90	2.88	2.73	2.69	2.68	2.71	2.80	2.82	2.67	2.64	2.61	2.62	2.69
3.00	2.65	2.55	2.50	2.50	2.56	2.62	2.60	2.47	2.43	2.44	2.45	2.52	2.54	2.41	2.38	2.36	2.36	2.43
3.25	2.43	2.33	2.27	2.28	2.33	2.39	2.38	2.24	2.21	2.22	2.22	2.29	2.32	2.18	2.15	2.13	2.14	2.21
3.50	2.22	2.13	2.08	2.08	2.13	2.17	2.17	2.05	2.01	2.02	2.04	2.09	2.12	1.99	1.98	1.94	1.94	2.01
3.75	2.04	1.95	1.91	1.91	1.96	2.01	1.99	1.88	1.85	1.85	1.86	1.92	1.94	1.83	1.81	1.79	1.80	1.85
4.00	1.88	1.80	1.76	1.76	1.80	1.84	1.83	1.73	1.71	1.70	1.72	1.77	1.79	1.69	1.66	1.65	1.65	1.70

k	0.10	0.20	0.25	0.30	0.40	0.50	0.10	0.20	0.25	0.30	0.40	0.50	0.10	0.20	0.25	0.30	0.40	0.50
λ	0.50	0.50	0.50	0.50	0.50	0.50	0.10	0.10	0.10	0.10	0.10	0.10	0.20	0.20	0.20	0.20	0.20	0.20
γ	1.50	1.50	1.50	1.50	1.50	1.50	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00
0.00	401	401	401	405	404	401	402	397	399	405	400	404	408	397	398	407	399	401
0.25	71.6	73.6	76.31	80.2	81.3	68.8	69.4	72.2	72.5	74.5	68.3	66.6	70.0	71.8	75.0	73.8	66.4	65.2
0.50	26.7	26.2	26.56	27.0	27.5	25.3	26.0	25.6	25.9	25.8	24.9	24.4	26.1	25.7	25.8	25.5	24.7	24.2
0.75	14.3	13.7	13.77	13.8	13.9	13.8	14.1	13.7	13.8	13.9	13.9	13.9	14.1	13.7	13.6	13.8	14.0	14.0
1.00	9.39	8.99	8.98	9.00	8.99	9.39	9.39	9.18	9.09	9.29	9.55	9.77	9.53	9.19	9.10	9.36	9.57	9.68
1.25	6.99	6.73	6.64	6.61	6.66	7.17	7.11	6.93	6.95	7.07	7.22	7.41	7.13	6.95	6.97	7.02	7.24	7.41
1.50	5.56	5.33	5.26	5.26	5.23	5.80	5.76	5.60	5.63	5.67	5.85	5.99	5.77	5.59	5.59	5.70	5.85	6.02
1.75	4.64	4.44	4.37	4.36	4.32	4.89	4.89	4.75	4.73	4.81	4.92	5.07	4.86	4.76	4.71	4.81	4.95	5.04
2.00	3.98	3.78	3.73	3.72	3.72	4.24	4.22	4.11	4.09	4.13	4.27	4.37	4.24	4.09	4.10	4.16	4.25	4.39
2.25	3.50	3.30	3.27	3.26	3.22	3.78	3.74	3.63	3.64	3.69	3.77	3.86	3.75	3.63	3.62	3.66	3.76	3.87
2.50	3.08	2.93	2.87	2.87	2.85	3.40	3.36	3.23	3.26	3.30	3.38	3.47	3.35	3.24	3.25	3.28	3.39	3.46
2.75	2.78	2.61	2.58	2.57	2.55	3.10	3.04	2.94	2.94	3.00	3.06	3.13	3.04	2.94	2.95	2.98	3.06	3.15
3.00	2.50	2.36	2.32	2.30	2.30	2.86	2.79	2.69	2.68	2.73	2.80	2.87	2.79	2.69	2.68	2.73	2.80	2.87
3.25	2.27	2.14	2.10	2.09	2.08	2.64	2.54	2.47	2.46	2.50	2.59	2.63	2.55	2.45	2.46	2.50	2.56	2.63
3.50	2.09	1.96	1.92	1.91	1.90	2.45	2.34	2.27	2.26	2.30	2.36	2.43	2.35	2.27	2.27	2.30	2.36	2.42
3.75	1.92	1.80	1.76	1.75	1.74	2.30	2.18	2.09	2.09	2.12	2.18	2.24	2.17	2.10	2.11	2.13	2.18	2.25
4.00	1.76	1.66	1.62	1.62	1.61	2.15	2.01	1.94	1.95	1.96	2.02	2.08	2.00	1.94	1.95	1.97	2.02	2.08

Table 3 (continued)

k	0.10	0.20	0.25	0.30	0.40	0.50	0.10	0.20	0.25	0.30	0.40	0.50	0.10	0.20	0.25	0.30	0.40	0.50
λ	0.30	0.30	0.30	0.30	0.30	0.30	0.40	0.40	0.40	0.40	0.40	0.40	0.50	0.50	0.50	0.50	0.50	0.50
γ	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00
0.00	398	406	403	403	403	404	405	397	398	403	403	405	400	400	404	401	407	405
0.25	68.7	71.9	76.0	78.1	70.2	67.0	69.6	71.5	74.8	75.9	73.0	69.3	70.2	72.0	76.4	79.4	75.8	70.8
0.50	26.1	25.9	25.7	26.0	25.3	24.4	26.4	25.6	25.9	26.3	25.9	24.9	26.3	25.8	26.3	26.3	26.7	25.3
0.75	14.2	13.8	13.7	13.8	13.9	14.1	14.2	13.7	13.6	13.8	13.8	14.0	14.3	13.8	13.7	13.8	14.0	14.0
1.00	9.48	9.19	9.12	9.21	9.37	9.60	9.50	9.15	9.02	9.08	9.23	9.48	9.43	9.08	8.97	9.08	9.17	9.37
1.25	7.07	6.91	6.85	6.93	7.09	7.34	7.13	6.79	6.76	6.84	6.90	7.19	7.03	6.79	6.75	6.69	6.85	6.98
1.50	5.74	5.55	5.55	5.57	5.69	5.93	5.71	5.49	5.43	5.46	5.56	5.80	5.66	5.42	5.39	5.37	5.39	5.59
1.75	4.81	4.68	4.65	4.70	4.82	4.99	4.82	4.61	4.56	4.56	4.67	4.80	4.72	4.51	4.48	4.45	4.52	4.65
2.00	4.18	4.04	4.02	4.04	4.13	4.29	4.14	3.95	3.93	3.93	4.00	4.15	4.06	3.89	3.86	3.85	3.87	3.99
2.25	3.69	3.59	3.54	3.58	3.67	3.77	3.68	3.48	3.44	3.47	3.50	3.62	3.57	3.38	3.36	3.35	3.38	3.49
2.50	3.28	3.20	3.17	3.19	3.26	3.38	3.26	3.10	3.07	3.09	3.12	3.24	3.22	3.00	2.99	2.98	2.99	3.10
2.75	2.98	2.89	2.87	2.89	2.94	3.04	2.93	2.79	2.76	2.75	2.81	2.89	2.87	2.69	2.68	2.67	2.68	2.76
3.00	2.71	2.62	2.60	2.62	2.67	2.76	2.66	2.51	2.49	2.52	2.53	2.61	2.58	2.45	2.41	2.40	2.41	2.49
3.25	2.48	2.40	2.38	2.38	2.46	2.53	2.42	2.30	2.28	2.27	2.32	2.40	2.35	2.22	2.20	2.18	2.19	2.26
3.50	2.28	2.21	2.18	2.21	2.23	2.32	2.22	2.09	2.08	2.10	2.12	2.18	2.15	2.02	2.02	1.99	2.00	2.06
3.75	2.09	2.02	2.00	2.01	2.07	2.13	2.03	1.92	1.91	1.92	1.95	2.01	1.97	1.87	1.84	1.83	1.84	1.89
4.00	1.94	1.87	1.85	1.87	1.91	1.97	1.88	1.77	1.76	1.76	1.79	1.86	1.82	1.71	1.70	1.69	1.70	1.75
k	0.10	0.20	0.25	0.30	0.40	0.50	0.10	0.20	0.25	0.30	0.40	0.50	0.10	0.20	0.25	0.30	0.40	0.50
λ	0.10	0.10	0.10	0.10	0.10	0.10	0.20	0.20	0.20	0.20	0.20	0.20	0.30	0.30	0.30	0.30	0.30	0.30
γ	2.50	2.50	2.50	2.50	2.50	2.50	2.50	2.50	2.50	2.50	2.50	2.50	2.50	2.50	2.50	2.50	2.50	2.50
0.00	401	400	404	396	407	397.5	396	402.4	402	407	403	403	398	398	404	399	396	401
0.25	69.2	71.9	71.4	66.8	65.0	64.2	67.8	72.7	72.2	69.8	65.5	65.0	68.9	71.8	74.3	72.1	65.2	64.9
0.50	25.3	25.1	25.0	24.4	24.3	24.1	25.6	25.3	25.3	24.9	24.5	24.2	25.9	25.4	25.6	25.3	24.4	24.4
0.75	13.8	13.6	13.8	13.8	14.2	14.00	13.8	13.68	13.7	13.8	14.1	14.0	14.0	13.7	13.8	13.7	13.9	14.0
1.00	9.36	9.31	9.48	9.56	9.85	9.79	9.35	9.26	9.34	9.51	9.74	9.81	9.42	9.22	9.27	9.34	9.54	9.74
1.25	7.13	7.13	7.24	7.34	7.57	7.54	7.13	7.05	7.10	7.23	7.46	7.54	7.14	6.98	7.02	7.08	7.27	7.46
1.50	5.77	5.78	5.88	5.96	6.14	6.17	5.77	5.73	5.76	5.88	6.05	6.14	5.78	5.66	5.67	5.72	5.91	6.05
1.75	4.89	4.88	4.96	5.04	5.18	5.19	4.89	4.85	4.88	4.94	5.11	5.17	4.87	4.76	4.80	4.87	4.97	5.10
2.00	4.26	4.27	4.32	4.36	4.50	4.52	4.22	4.21	4.23	4.30	4.44	4.50	4.24	4.13	4.15	4.20	4.33	4.44
2.25	3.78	3.76	3.83	3.87	4.00	4.01	3.76	3.73	3.75	3.81	3.93	4.00	3.73	3.65	3.69	3.71	3.81	3.92
2.50	3.40	3.40	3.45	3.49	3.59	3.62	3.38	3.36	3.39	3.43	3.54	3.60	3.37	3.29	3.31	3.35	3.44	3.52
2.75	3.10	3.11	3.14	3.17	3.28	3.28	3.09	3.06	3.08	3.12	3.22	3.26	3.07	2.98	3.01	3.04	3.11	3.19
3.00	2.85	2.85	2.89	2.93	3.01	3.04	2.82	2.80	2.83	2.87	2.96	3.01	2.80	2.73	2.75	2.78	2.85	2.92
3.25	2.64	2.64	2.68	2.71	2.80	2.81	2.61	2.59	2.61	2.64	2.73	2.77	2.57	2.52	2.53	2.56	2.62	2.69
3.50	2.45	2.46	2.49	2.52	2.60	2.61	2.43	2.41	2.42	2.46	2.53	2.58	2.38	2.32	2.33	2.37	2.42	2.49
3.75	2.29	2.28	2.32	2.35	2.42	2.43	2.25	2.23	2.25	2.28	2.36	2.39	2.21	2.16	2.16	2.18	2.25	2.31
4.00	2.14	2.15	2.17	2.20	2.26	2.27	2.10	2.08	2.10	2.13	2.19	2.23	2.04	2.00	2.02	2.03	2.08	2.14

Table 3 (continued)

k	0.10	0.20	0.25	0.30	0.40	0.50	0.10	0.20	0.25	0.30	0.40	0.50	0.10	0.20	0.25	0.30	0.40	0.50
λ	0.40	0.40	0.40	0.40	0.40	0.40	0.50	0.50	0.50	0.50	0.50	0.50	0.10	0.10	0.10	0.10	0.10	0.10
γ	2.50	2.50	2.50	2.50	2.50	2.50	2.50	2.50	2.50	2.50	2.50	2.50	3.00	3.00	3.00	3.00	3.00	3.00
0.00	406	403	397	396	398	407	406	403	406	405	404	395	402	404	396	403	406	399
0.25	69.4	72.0	73.7	74.6	68.5	65.6	69.1	73.3	75.8	77.3	72.7	67.5	69.4	71.5	67.3	65.4	65.3	64.8
0.50	26.2	25.6	25.7	25.7	24.9	24.5	26.5	25.8	26.0	26.3	25.7	24.7	25.3	25.0	24.5	24.6	24.3	24.2
0.75	14.0	13.7	13.7	13.7	13.9	14.1	14.1	13.8	13.8	13.7	13.8	13.8	13.8	13.8	13.8	14.1	14.2	14.1
1.00	9.48	9.19	9.12	9.25	9.42	9.65	9.48	9.14	9.14	9.08	9.29	9.52	9.39	9.50	9.64	9.80	9.85	9.91
1.25	7.12	6.91	6.92	6.93	7.16	7.39	7.12	6.85	6.81	6.80	6.98	7.15	7.17	7.29	7.40	7.52	7.61	7.56
1.50	5.77	5.61	5.56	5.60	5.80	5.97	5.70	5.51	5.49	5.49	5.64	5.76	5.85	5.96	6.01	6.15	6.21	6.20
1.75	4.87	4.71	4.70	4.72	4.86	5.03	4.81	4.63	4.62	4.61	4.69	4.83	4.95	5.04	5.07	5.19	5.24	5.26
2.00	4.22	4.08	4.06	4.09	4.20	4.34	4.16	4.01	3.98	3.95	4.05	4.15	4.30	4.37	4.42	4.51	4.57	4.55
2.25	3.72	3.61	3.58	3.61	3.70	3.83	3.66	3.51	3.50	3.48	3.56	3.65	3.81	3.88	3.93	4.00	4.03	4.03
2.50	3.33	3.22	3.21	3.22	3.33	3.43	3.27	3.12	3.12	3.11	3.16	3.23	3.44	3.49	3.52	3.59	3.64	3.64
2.75	3.02	2.91	2.90	2.92	3.00	3.09	2.95	2.82	2.81	2.80	2.84	2.91	3.14	3.18	3.22	3.28	3.32	3.32
3.00	2.75	2.66	2.64	2.66	2.73	2.82	2.68	2.55	2.54	2.53	2.57	2.64	2.89	2.94	2.97	3.02	3.06	3.06
3.25	2.53	2.42	2.42	2.45	2.50	2.58	2.43	2.33	2.32	2.31	2.36	2.41	2.68	2.71	2.76	2.80	2.83	2.83
3.50	2.33	2.24	2.22	2.24	2.31	2.37	2.24	2.14	2.12	2.12	2.16	2.21	2.50	2.54	2.56	2.61	2.64	2.64
3.75	2.14	2.07	2.05	2.06	2.12	2.19	2.06	1.96	1.94	1.94	1.98	2.03	2.34	2.37	2.39	2.44	2.46	2.47
4.00	1.98	1.90	1.90	1.91	1.97	2.03	1.90	1.80	1.80	1.79	1.83	1.88	2.20	2.24	2.25	2.30	2.32	2.32

k	0.10	0.20	0.25	0.30	0.40	0.50	0.10	0.20	0.25	0.30	0.40	0.50	0.10	0.20	0.25	0.30	0.40	0.50
λ	0.20	0.20	0.20	0.20	0.20	0.20	0.30	0.30	0.30	0.30	0.30	0.30	0.40	0.40	0.40	0.40	0.40	0.40
γ	3.00	3.00	3.00	3.00	3.00	3.00	3.00	3.00	3.00	3.00	3.00	3.00	3.00	3.00	3.00	3.00	3.00	3.00
0.00	395	397.1	397	401	396	400	399	397	402	396	401	402	400	404	401	398	399	408
0.25	69.0	71.3	69.1	66.2	64.3	64.0	68.2	71.5	71.2	68.2	65.0	63.6	67.7	72.2	72.1	71.2	66.2	65.6
0.50	25.5	24.9	24.8	24.4	24.4	24.4	25.6	25.0	25.4	24.6	24.5	24.3	25.8	25.4	25.3	25.3	24.4	24.5
0.75	13.8	13.68	13.7	14.0	14.0	14.0	13.9	13.6	13.7	13.8	14.0	14.0	13.9	13.7	13.6	13.8	14.0	14.1
1.00	9.39	9.38	9.49	9.69	9.79	9.82	9.37	9.32	9.38	9.45	9.78	9.79	9.36	9.23	9.19	9.40	9.60	9.85
1.25	7.19	7.18	7.27	7.41	7.53	7.60	7.13	7.09	7.14	7.26	7.42	7.54	7.12	7.02	7.01	7.12	7.35	7.55
1.50	5.82	5.83	5.93	6.01	6.14	6.20	5.78	5.76	5.79	5.87	6.10	6.13	5.76	5.68	5.68	5.78	5.95	6.09
1.75	4.93	4.92	5.01	5.08	5.18	5.20	4.88	4.86	4.90	4.97	5.13	5.19	4.88	4.77	4.78	4.87	5.03	5.14
2.00	4.27	4.28	4.35	4.43	4.51	4.54	4.24	4.22	4.27	4.31	4.47	4.50	4.22	4.16	4.15	4.22	4.37	4.47
2.25	3.78	3.79	3.86	3.93	4.00	4.02	3.75	3.75	3.78	3.83	3.97	4.00	3.75	3.68	3.67	3.74	3.84	3.96
2.50	3.42	3.43	3.48	3.54	3.61	3.63	3.38	3.37	3.41	3.45	3.56	3.59	3.36	3.31	3.32	3.36	3.48	3.55
2.75	3.12	3.13	3.17	3.23	3.28	3.31	3.09	3.07	3.10	3.15	3.24	3.27	3.05	3.01	3.01	3.05	3.15	3.23
3.00	2.87	2.86	2.92	2.97	3.02	3.04	2.84	2.83	2.85	2.89	2.99	3.01	2.80	2.77	2.75	2.80	2.88	2.96
3.25	2.65	2.66	2.71	2.75	2.79	2.82	2.63	2.61	2.64	2.67	2.76	2.79	2.59	2.54	2.55	2.57	2.66	2.73
3.50	2.46	2.48	2.51	2.56	2.60	2.63	2.44	2.41	2.46	2.47	2.57	2.59	2.39	2.35	2.34	2.38	2.46	2.52
3.75	2.30	2.31	2.34	2.39	2.43	2.45	2.27	2.25	2.28	2.31	2.38	2.41	2.21	2.18	2.18	2.21	2.28	2.33
4.00	2.17	2.17	2.20	2.24	2.28	2.30	2.12	2.11	2.13	2.15	2.23	2.24	2.05	2.02	2.02	2.05	2.11	2.17

Table 3 (continued)

k	0.10	0.20	0.25	0.30	0.40	0.50	0.10	0.20	0.25	0.30	0.40	0.50	0.10	0.20	0.25	0.30	0.40	0.50
λ	0.50	0.50	0.50	0.50	0.50	0.50	0.10	0.10	0.10	0.10	0.10	0.10	0.20	0.20	0.20	0.20	0.20	0.20
γ	3.00	3.00	3.00	3.00	3.00	3.00	4.00	4.00	4.00	4.00	4.00	4.00	4.00	4.00	4.00	4.00	4.00	4.00
0.00	405	400	399	399	397	402	399	406	405	400	397	405	402	398	400	401	396	401
0.25	69.9	73.0	73.4	74.1	68.7	66.1	69.4	66.7	64.7	64.3	64.8	65.1	68.8	67.9	65.3	66.0	63.8	65.2
0.50	26.0	25.6	25.6	25.8	24.9	24.5	24.6	24.7	24.5	24.2	24.2	24.3	24.5	24.7	24.4	24.1	24.4	24.2
0.75	14.1	13.7	13.7	13.7	13.8	14.0	13.8	13.9	14.2	14.0	14.0	14.1	13.7	13.8	14.1	14.0	14.1	14.2
1.00	9.47	9.23	9.14	9.21	9.44	9.69	9.55	9.81	9.81	9.86	9.87	9.82	9.41	9.64	9.84	9.86	9.86	9.83
1.25	7.14	6.94	6.90	6.95	7.12	7.34	7.33	7.53	7.61	7.61	7.61	7.65	7.25	7.43	7.60	7.57	7.60	7.52
1.50	5.76	5.62	5.58	5.62	5.76	5.94	6.00	6.17	6.22	6.19	6.21	6.19	5.96	6.08	6.16	6.20	6.20	6.21
1.75	4.86	4.72	4.67	4.73	4.86	5.00	5.07	5.23	5.25	5.26	5.25	5.24	5.01	5.14	5.20	5.21	5.24	5.24
2.00	4.22	4.07	4.05	4.09	4.20	4.33	4.43	4.52	4.58	4.55	4.57	4.57	4.36	4.46	4.51	4.56	4.56	4.57
2.25	3.71	3.61	3.59	3.61	3.72	3.82	3.91	4.03	4.03	4.04	4.06	4.03	3.87	3.96	4.01	4.04	4.05	4.04
2.50	3.34	3.23	3.20	3.22	3.32	3.42	3.54	3.61	3.64	3.64	3.64	3.64	3.50	3.56	3.62	3.64	3.66	3.65
2.75	3.02	2.93	2.90	2.93	2.99	3.09	3.21	3.30	3.32	3.30	3.32	3.31	3.17	3.26	3.31	3.32	3.31	3.31
3.00	2.75	2.66	2.64	2.66	2.75	2.82	2.98	3.04	3.06	3.06	3.06	3.07	2.93	2.99	3.03	3.06	3.06	3.07
3.25	2.53	2.44	2.41	2.44	2.50	2.57	2.75	2.82	2.84	2.84	2.84	2.84	2.71	2.78	2.82	2.83	2.85	2.84
3.50	2.33	2.24	2.23	2.24	2.30	2.37	2.55	2.63	2.65	2.65	2.65	2.64	2.53	2.58	2.64	2.65	2.65	2.64
3.75	2.14	2.07	2.05	2.07	2.13	2.19	2.40	2.46	2.48	2.47	2.47	2.49	2.37	2.42	2.45	2.46	2.47	2.47
4.00	1.98	1.92	1.90	1.91	1.97	2.03	2.28	2.31	2.33	2.32	2.33	2.33	2.23	2.28	2.32	2.33	2.33	2.33

k	0.10	0.20	0.25	0.30	0.40	0.50	0.10	0.20	0.25	0.30	0.40	0.50	0.10	0.20	0.25	0.30	0.40	0.50
λ	0.30	0.30	0.30	0.30	0.30	0.30	0.40	0.40	0.40	0.40	0.40	0.40	0.50	0.50	0.50	0.50	0.50	0.50
γ	4.00	4.00	4.00	4.00	4.00	4.00	4.00	4.00	4.00	4.00	4.00	4.00	4.00	4.00	4.00	4.00	4.00	4.00
0.00	399	397	404	406	405	399	406	408	408	403	398	402	401	406	407	405	407	407
0.25	68.2	70.1	66.4	65.6	65.2	64.6	69.6	72.6	70.7	65.9	64.8	64.5	68.6	73.5	71.1	69.4	65.9	65.0
0.50	24.6	24.4	24.1	24.8	24.3	24.3	25.0	25.0	25.2	24.6	24.4	24.4	25.3	25.1	25.1	25.0	24.4	24.2
0.75	13.6	13.8	13.8	14.1	13.9	14.0	13.9	13.7	13.8	13.9	13.9	14.0	13.9	13.7	13.6	13.8	13.9	14.0
1.00	9.41	9.50	9.61	9.86	9.88	9.86	9.39	9.45	9.55	9.66	9.78	9.89	9.35	9.30	9.34	9.48	9.63	9.90
1.25	7.17	7.31	7.40	7.57	7.57	7.58	7.19	7.15	7.28	7.43	7.52	7.57	7.14	7.03	7.11	7.20	7.42	7.59
1.50	5.83	5.98	6.05	6.17	6.21	6.19	5.78	5.85	5.95	6.03	6.14	6.22	5.79	5.71	5.77	5.89	6.06	6.14
1.75	4.94	5.04	5.09	5.24	5.28	5.24	4.92	4.94	5.01	5.10	5.18	5.23	4.90	4.85	4.87	4.93	5.12	5.20
2.00	4.31	4.37	4.44	4.52	4.54	4.55	4.27	4.29	4.34	4.43	4.52	4.52	4.23	4.23	4.23	4.33	4.44	4.49
2.25	3.81	3.88	3.93	4.03	4.04	4.03	3.80	3.82	3.87	3.94	4.00	4.05	3.76	3.72	3.75	3.82	3.93	3.99
2.50	3.43	3.50	3.54	3.61	3.63	3.63	3.42	3.44	3.49	3.55	3.59	3.62	3.38	3.35	3.37	3.44	3.55	3.60
2.75	3.13	3.20	3.24	3.30	3.31	3.32	3.11	3.13	3.17	3.21	3.30	3.32	3.08	3.06	3.08	3.12	3.23	3.27
3.00	2.87	2.94	2.97	3.04	3.07	3.07	2.88	2.90	2.93	2.97	3.02	3.05	2.84	2.80	2.83	2.87	2.95	3.01
3.25	2.69	2.74	2.75	2.83	2.84	2.85	2.66	2.66	2.69	2.75	2.80	2.82	2.62	2.60	2.61	2.65	2.73	2.77
3.50	2.50	2.55	2.57	2.63	2.64	2.64	2.47	2.49	2.52	2.55	2.61	2.63	2.42	2.40	2.41	2.44	2.53	2.58
3.75	2.34	2.37	2.41	2.45	2.47	2.46	2.31	2.32	2.36	2.39	2.43	2.46	2.25	2.23	2.24	2.29	2.36	2.39
4.00	2.20	2.24	2.26	2.31	2.32	2.32	2.18	2.20	2.21	2.24	2.27	2.31	2.10	2.08	2.10	2.13	2.21	2.23

Table 4 ARLs properties of proposed $\text{ACUSUM}_c^{(2)}$ control chart at different values of k, λ, and γ when $\text{ARL}_0 = 400$ and $\delta_{\min}^{+} = 0.50$

k	0.10	0.20	0.25	0.30	0.40	0.50	0.10	0.20	0.25	0.30	0.40	0.50	0.10	0.20	0.25	0.30	0.40	0.50
λ	0.10	0.10	0.10	0.10	0.10	0.10	0.20	0.20	0.20	0.20	0.20	0.20	0.30	0.30	0.30	0.30	0.30	0.30
γ	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
0.00	395	399	398	399	399	403	398	398.2	402.5	405.2	406	406	407	397	406	402	406.1	399
0.25	70.7	67.1	65.8	64.7	64.3	65.6	70.6	67.3	65.8	65.2	65.5	65.9	70.7	66.9	66.1	64.4	65.1	66.1
0.50	24.7	25.7	25.9	26.2	25.9	25.8	24.5	25.6	26.1	26.0	26.0	26.0	24.7	25.6	26.1	25.9	25.9	25.7
0.75	14.1	14.9	15.2	15.6	15.6	15.5	14.1	15.04	15.31	15.57	15.6	15.5	14.2	15.0	15.4	15.4	15.54	15.4
1.00	9.69	10.3	10.6	10.8	11.03	11.08	9.67	10.34	10.62	10.86	11.07	11.03	9.71	10.38	10.74	10.80	10.98	10.98
1.25	7.26	7.80	7.99	8.21	8.41	8.53	7.33	7.82	8.07	8.26	8.42	8.46	7.35	7.85	8.09	8.24	8.44	8.35
1.50	5.80	6.20	6.41	6.57	6.75	6.83	5.80	6.20	6.44	6.58	6.78	6.84	5.86	6.25	6.47	6.54	6.73	6.77
1.75	4.77	5.12	5.28	5.47	5.60	5.69	4.79	5.14	5.33	5.42	5.61	5.67	4.82	5.12	5.34	5.42	5.58	5.60
2.00	4.06	4.32	4.46	4.59	4.75	4.84	4.03	4.33	4.50	4.59	4.75	4.82	4.06	4.34	4.48	4.59	4.73	4.77
2.25	3.50	3.74	3.85	3.97	4.10	4.19	3.48	3.76	3.88	3.97	4.11	4.16	3.51	3.74	3.90	3.96	4.09	4.13
2.50	3.06	3.29	3.37	3.47	3.59	3.66	3.07	3.28	3.39	3.49	3.59	3.67	3.08	3.27	3.42	3.48	3.60	3.63
2.75	2.72	2.92	3.00	3.10	3.20	3.25	2.73	2.92	3.03	3.10	3.20	3.26	2.75	2.93	3.04	3.09	3.19	3.23
3.00	2.46	2.64	2.72	2.77	2.87	2.94	2.47	2.64	2.72	2.78	2.89	2.95	2.48	2.65	2.74	2.79	2.88	2.91
3.25	2.26	2.40	2.48	2.54	2.64	2.68	2.25	2.41	2.49	2.54	2.63	2.68	2.27	2.42	2.51	2.53	2.63	2.66
3.50	2.08	2.23	2.28	2.36	2.42	2.46	2.09	2.22	2.29	2.35	2.43	2.46	2.11	2.23	2.31	2.35	2.41	2.46
3.75	1.94	2.07	2.13	2.18	2.25	2.29	1.95	2.07	2.14	2.19	2.26	2.29	1.96	2.08	2.15	2.18	2.25	2.28
4.00	1.81	1.94	1.99	2.05	2.12	2.15	1.82	1.94	2.00	2.06	2.12	2.16	1.84	1.95	2.02	2.06	2.12	2.14

k	0.10	0.20	0.25	0.30	0.40	0.50	0.10	0.20	0.25	0.30	0.40	0.50	0.10	0.20	0.25	0.30	0.40	0.50
λ	0.40	0.40	0.40	0.40	0.40	0.40	0.50	0.50	0.50	0.50	0.50	0.50	0.10	0.10	0.10	0.10	0.10	0.10
γ	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.50	1.50	1.50	1.50	1.50	1.50
0.00	404	402	398	401	401	397	404	397.6	399.2	397	405	403	398	399	404	407	397	402
0.25	71.2	67.8	65.0	64.7	65.0	65.5	72.4	66.5	65.0	64.0	65.0	66.7	70.12	67.98	67.90	66.55	67.87	68.70
0.50	24.9	26.0	25.9	26.1	25.8	25.4	24.8	25.6	26.1	25.8	25.8	25.6	24.66	25.28	25.75	25.58	25.58	25.52
0.75	14.2	15.2	15.2	15.5	15.5	15.3	14.1	14.98	15.30	15.5	15.5	15.2	14.14	14.78	14.96	14.94	14.76	14.77
1.00	9.78	10.43	10.63	10.88	10.97	10.80	9.74	10.43	10.68	10.84	10.92	10.71	9.68	10.22	10.40	10.38	10.27	10.17
1.25	7.35	7.89	8.07	8.24	8.38	8.33	7.36	7.86	8.11	8.26	8.34	8.27	7.30	7.71	7.85	7.83	7.78	7.74
1.50	5.85	6.28	6.46	6.58	6.71	6.65	5.83	6.26	6.44	6.61	6.70	6.67	5.80	6.12	6.24	6.26	6.23	6.23
1.75	4.81	5.16	5.32	5.43	5.58	5.54	4.78	5.15	5.32	5.45	5.53	5.55	4.79	5.05	5.14	5.17	5.16	5.16
2.00	4.05	4.37	4.49	4.60	4.72	4.72	4.04	4.36	4.52	4.61	4.69	4.69	4.06	4.25	4.37	4.39	4.39	4.36
2.25	3.50	3.76	3.87	3.96	4.08	4.07	3.49	3.76	3.89	3.99	4.06	4.07	3.50	3.70	3.77	3.79	3.81	3.80
2.50	3.09	3.31	3.41	3.48	3.58	3.60	3.07	3.30	3.41	3.50	3.56	3.58	3.09	3.26	3.32	3.33	3.34	3.34
2.75	2.74	2.95	3.03	3.10	3.18	3.20	2.74	2.95	3.03	3.11	3.16	3.19	2.75	2.89	2.95	2.98	2.98	3.00
3.00	2.48	2.65	2.73	2.80	2.88	2.88	2.48	2.66	2.74	2.80	2.86	2.89	2.48	2.61	2.68	2.70	2.69	2.70
3.25	2.27	2.43	2.49	2.55	2.62	2.64	2.27	2.43	2.50	2.56	2.61	2.62	2.27	2.40	2.44	2.44	2.47	2.48
3.50	2.10	2.25	2.31	2.36	2.41	2.43	2.11	2.25	2.32	2.35	2.41	2.42	2.09	2.21	2.25	2.26	2.28	2.28
3.75	1.96	2.10	2.15	2.20	2.25	2.26	1.97	2.10	2.16	2.20	2.25	2.26	1.94	2.05	2.09	2.10	2.11	2.12
4.00	1.84	1.98	2.02	2.06	2.11	2.13	1.85	1.97	2.03	2.08	2.12	2.13	1.80	1.91	1.95	1.96	1.98	1.99

Table 4 (continued)

k	0.10	0.20	0.25	0.30	0.40	0.50	0.10	0.20	0.25	0.30	0.40	0.50	0.10	0.20	0.25	0.30	0.40	0.50
λ	0.20	0.20	0.20	0.20	0.20	0.20	0.30	0.30	0.30	0.30	0.30	0.30	0.40	0.40	0.40	0.40	0.40	0.40
γ	1.50	1.50	1.50	1.50	1.50	1.50	1.50	1.50	1.50	1.50	1.50	1.50	1.50	1.50	1.50	1.50	1.50	1.50
0.00	401	400.6	402.0	403	401	403	400	397	403	395	404.0	399	407	400	407	398	396	397
0.25	70.9	67.1	67.2	66.9	67.6	68.5	71.3	68.3	66.7	66.4	67.9	69.6	71.7	67.58	66.8	67.9	68.7	69.3
0.50	24.6	25.4	25.5	25.6	25.4	25.3	24.7	25.5	25.6	25.5	25.4	25.7	25.0	25.34	25.7	25.5	25.6	25.6
0.75	14.2	14.79	14.94	14.9	14.7	14.6	14.1	14.8	15.0	15.0	14.75	14.7	14.1	14.74	15.1	14.7	14.7	14.6
1.00	9.72	10.21	10.33	10.31	10.24	10.14	9.75	10.18	10.42	10.36	10.26	10.14	9.81	10.28	10.40	10.20	10.22	9.95
1.25	7.30	7.65	7.86	7.84	7.77	7.71	7.34	7.66	7.84	7.84	7.78	7.69	7.37	7.74	7.89	7.69	7.73	7.61
1.50	5.83	6.10	6.23	6.25	6.22	6.17	5.80	6.12	6.23	6.24	6.21	6.17	5.81	6.15	6.30	6.15	6.16	6.10
1.75	4.79	5.06	5.14	5.16	5.15	5.12	4.80	5.05	5.16	5.15	5.14	5.12	4.80	5.06	5.14	5.12	5.12	5.06
2.00	4.06	4.28	4.35	4.37	4.37	4.33	4.08	4.26	4.36	4.38	4.37	4.35	4.07	4.28	4.37	4.35	4.35	4.31
2.25	3.50	3.69	3.75	3.79	3.78	3.75	3.53	3.72	3.78	3.81	3.78	3.79	3.52	3.71	3.80	3.77	3.78	3.75
2.50	3.08	3.26	3.31	3.33	3.35	3.34	3.08	3.26	3.32	3.33	3.34	3.32	3.09	3.26	3.34	3.32	3.32	3.30
2.75	2.76	2.92	2.96	2.98	2.98	2.97	2.76	2.90	2.97	2.99	2.99	2.98	2.77	2.93	2.98	2.97	2.97	2.96
3.00	2.50	2.62	2.67	2.69	2.69	2.69	2.50	2.63	2.68	2.70	2.70	2.70	2.51	2.65	2.70	2.70	2.70	2.68
3.25	2.28	2.39	2.44	2.46	2.46	2.47	2.28	2.40	2.45	2.47	2.47	2.47	2.29	2.40	2.46	2.48	2.46	2.46
3.50	2.10	2.21	2.26	2.28	2.28	2.27	2.11	2.22	2.27	2.28	2.28	2.28	2.12	2.23	2.28	2.28	2.28	2.26
3.75	1.94	2.05	2.09	2.10	2.12	2.12	1.96	2.08	2.11	2.12	2.12	2.13	1.98	2.09	2.13	2.13	2.13	2.11
4.00	1.82	1.91	1.96	1.98	1.98	1.99	1.83	1.93	1.97	1.99	1.998	2.00	1.85	1.95	1.99	1.99	1.99	1.99
k	0.10	0.20	0.25	0.30	0.40	0.50	0.10	0.20	0.25	0.30	0.40	0.50	0.10	0.20	0.25	0.30	0.40	0.50
λ	0.50	0.50	0.50	0.50	0.50	0.50	0.10	0.10	0.10	0.10	0.10	0.10	0.20	0.20	0.20	0.20	0.20	0.20
γ	1.50	1.50	1.50	1.50	1.50	1.50	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00
0.00	402	397.2	399.2	401	408	402	399	406	403	397	399	396	403	400.2	395.7	396	397	402
0.25	71.1	66.5	66.9	66.0	69.4	69.3	69.6	68.1	67.2	68.4	72.3	72.8	69.6	67.6	65.9	69.7	71.7	72.9
0.50	24.8	25.6	25.9	25.4	25.8	25.6	24.8	25.5	25.5	25.3	26.0	26.0	25.0	25.4	25.3	25.4	25.6	26.2
0.75	14.2	14.91	15.07	14.9	14.7	14.4	14.3	14.8	14.7	14.6	14.4	14.3	14.2	14.67	14.61	14.5	14.2	14.4
1.00	9.74	10.26	10.39	10.36	10.23	9.92	9.76	10.1	10.1	9.9	9.73	9.63	9.81	10.07	10.01	9.91	9.68	9.66
1.25	7.34	7.73	7.86	7.84	7.77	7.50	7.38	7.64	7.57	7.48	7.32	7.21	7.31	7.56	7.55	7.53	7.29	7.21
1.50	5.78	6.17	6.26	6.27	6.24	6.04	5.83	6.06	6.08	5.97	5.84	5.75	5.82	6.05	6.00	6.01	5.84	5.78
1.75	4.78	5.09	5.21	5.20	5.16	5.02	4.82	5.02	5.00	4.94	4.83	4.75	4.80	4.99	4.98	4.94	4.82	4.75
2.00	4.06	4.29	4.40	4.39	4.38	4.26	4.07	4.24	4.24	4.21	4.12	4.03	4.09	4.23	4.21	4.21	4.09	4.03
2.25	3.50	3.70	3.79	3.82	3.81	3.71	3.55	3.69	3.68	3.63	3.56	3.50	3.54	3.66	3.66	3.64	3.54	3.49
2.50	3.10	3.27	3.34	3.35	3.36	3.28	3.11	3.25	3.24	3.20	3.14	3.09	3.10	3.22	3.23	3.21	3.13	3.10
2.75	2.76	2.93	2.99	2.99	3.00	2.94	2.76	2.90	2.89	2.87	2.80	2.75	2.78	2.88	2.87	2.87	2.79	2.76
3.00	2.51	2.65	2.70	2.69	2.71	2.66	2.51	2.61	2.59	2.59	2.52	2.49	2.52	2.61	2.61	2.60	2.53	2.50
3.25	2.30	2.43	2.47	2.48	2.48	2.45	2.28	2.39	2.38	2.35	2.31	2.26	2.29	2.37	2.37	2.37	2.31	2.27
3.50	2.13	2.25	2.29	2.30	2.30	2.26	2.09	2.19	2.18	2.16	2.12	2.09	2.10	2.18	2.18	2.18	2.13	2.10
3.75	1.99	2.09	2.13	2.14	2.14	2.12	1.93	2.02	2.02	2.00	1.96	1.93	1.94	2.02	2.01	2.02	1.95	1.94
4.00	1.86	1.97	2.00	2.02	2.03	2.00	1.79	1.88	1.88	1.86	1.82	1.79	1.80	1.8807	1.8827	1.87	1.82	1.81

Table 4 (continued)

k	0.10	0.20	0.25	0.30	0.40	0.50	0.10	0.20	0.25	0.30	0.40	0.50	0.10	0.20	0.25	0.30	0.40	0.50
λ	0.30	0.30	0.30	0.30	0.30	0.30	0.40	0.40	0.40	0.40	0.40	0.40	0.50	0.50	0.50	0.50	0.50	0.50
γ	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00
0.00	402	404	402	401	405.8	398	396	407	407	403	399	404	399	398.6	401.4	403	405	400
0.25	69.7	67.4	67.0	69.0	72.8	74.0	69.6	68.0	67.5	69.1	71.8	74.0	70.6	66.4	66.5	67.9	71.5	73.6
0.50	25.0	25.6	25.5	25.5	25.9	26.0	24.7	25.8	25.4	25.4	25.6	26.2	25.0	25.4	25.4	25.4	25.7	25.8
0.75	14.3	14.7	14.8	14.6	14.24	14.2	14.2	14.9	14.7	14.6	14.1	14.2	14.3	14.71	14.74	14.6	14.2	13.9
1.00	9.85	10.09	10.18	9.97	9.70	9.54	9.70	10.18	10.16	9.96	9.64	9.53	9.84	10.11	10.12	9.97	9.65	9.42
1.25	7.33	7.64	7.64	7.54	7.32	7.16	7.32	7.66	7.64	7.52	7.23	7.13	7.36	7.64	7.59	7.56	7.25	7.02
1.50	5.84	6.05	6.08	5.99	5.81	5.72	5.80	6.10	6.10	6.00	5.76	5.67	5.80	6.06	6.05	6.00	5.80	5.62
1.75	4.80	5.00	5.02	4.97	4.83	4.71	4.79	5.05	4.99	4.97	4.77	4.69	4.83	4.99	5.01	4.97	4.80	4.64
2.00	4.10	4.24	4.25	4.23	4.11	4.01	4.05	4.29	4.26	4.23	4.07	4.00	4.08	4.23	4.26	4.23	4.08	3.96
2.25	3.54	3.69	3.69	3.66	3.56	3.50	3.51	3.71	3.70	3.65	3.54	3.48	3.54	3.69	3.70	3.67	3.55	3.45
2.50	3.12	3.24	3.25	3.23	3.15	3.08	3.11	3.27	3.26	3.23	3.13	3.08	3.12	3.24	3.26	3.25	3.15	3.05
2.75	2.79	2.90	2.90	2.88	2.81	2.75	2.78	2.93	2.92	2.89	2.80	2.76	2.81	2.92	2.92	2.90	2.82	2.75
3.00	2.52	2.62	2.63	2.61	2.56	2.49	2.52	2.65	2.64	2.63	2.53	2.50	2.53	2.64	2.65	2.63	2.56	2.49
3.25	2.30	2.39	2.40	2.38	2.32	2.27	2.31	2.43	2.42	2.40	2.32	2.28	2.32	2.41	2.43	2.41	2.34	2.28
3.50	2.11	2.20	2.21	2.19	2.14	2.10	2.13	2.24	2.23	2.21	2.14	2.12	2.15	2.24	2.26	2.23	2.17	2.11
3.75	1.96	2.04	2.05	2.03	1.99	1.94	1.95	2.08	2.07	2.06	1.99	1.95	2.00	2.07	2.09	2.08	2.02	1.97
4.00	1.82	1.90	1.91	1.90	1.84	1.82	1.83	1.94	1.93	1.93	1.85	1.83	1.87	1.9412	1.9546	1.94	1.89	1.84
k	0.10	0.20	0.25	0.30	0.40	0.50	0.10	0.20	0.25	0.30	0.40	0.50	0.10	0.20	0.25	0.30	0.40	0.50
λ	0.10	0.10	0.10	0.10	0.10	0.10	0.20	0.20	0.20	0.20	0.20	0.20	0.30	0.30	0.30	0.30	0.30	0.30
γ	2.50	2.50	2.50	2.50	2.50	2.50	2.50	2.50	2.50	2.50	2.50	2.50	2.50	2.50	2.50	2.50	2.50	2.50
0.00	399	405	401	402	399	396	398	396.8	399.6	403	403.9	404	403	406	406	403	398.4	399
0.25	68.1	68.1	68.4	70.5	74.7	76.5	69.0	66.3	67.1	70.4	75.9	75.9	69.4	67.1	70.2	70.2	75.6	77.9
0.50	25.3	25.4	25.4	25.5	26.0	26.7	24.9	25.4	25.2	25.8	26.3	26.3	25.2	25.6	25.5	25.6	26.4	27.3
0.75	14.3	14.6	14.5	14.3	14.1	14.3	14.4	14.67	14.50	14.2	13.94	13.9	14.5	14.6	14.2	14.3	14.04	14.2
1.00	9.83	10.0	9.9	9.7	9.42	9.35	9.82	10.00	9.89	9.69	9.41	9.41	9.83	9.99	9.70	9.69	9.32	9.27
1.25	7.36	7.55	7.42	7.30	6.97	6.92	7.40	7.47	7.45	7.29	6.99	6.99	7.41	7.55	7.23	7.28	6.95	6.85
1.50	5.87	5.99	5.94	5.81	5.56	5.46	5.89	5.99	5.90	5.80	5.55	5.55	5.89	5.97	5.81	5.82	5.53	5.43
1.75	4.86	4.95	4.92	4.82	4.59	4.51	4.85	4.94	4.88	4.79	4.60	4.60	4.88	4.95	4.80	4.80	4.56	4.46
2.00	4.12	4.20	4.15	4.09	3.87	3.81	4.13	4.20	4.15	4.09	3.90	3.90	4.11	4.21	4.09	4.06	3.88	3.79
2.25	3.57	3.65	3.62	3.54	3.36	3.28	3.57	3.62	3.61	3.55	3.37	3.37	3.57	3.65	3.55	3.54	3.37	3.29
2.50	3.14	3.22	3.19	3.11	2.95	2.88	3.13	3.21	3.17	3.13	2.96	2.96	3.15	3.23	3.11	3.11	2.96	2.87
2.75	2.79	2.85	2.84	2.77	2.63	2.56	2.80	2.86	2.83	2.78	2.63	2.63	2.81	2.88	2.77	2.78	2.64	2.56
3.00	2.52	2.57	2.54	2.48	2.35	2.29	2.53	2.58	2.55	2.50	2.36	2.36	2.55	2.60	2.50	2.51	2.37	2.31
3.25	2.28	2.33	2.31	2.25	2.14	2.09	2.29	2.34	2.31	2.28	2.14	2.14	2.31	2.37	2.29	2.28	2.15	2.10
3.50	2.08	2.13	2.11	2.06	1.95	1.90	2.10	2.15	2.11	2.07	1.96	1.96	2.12	2.16	2.08	2.08	1.98	1.92
3.75	1.92	1.96	1.95	1.90	1.79	1.75	1.93	1.97	1.95	1.91	1.80	1.80	1.94	2.00	1.93	1.93	1.82	1.76
4.00	1.77	1.81	1.79	1.75	1.65	1.61	1.79	1.8213	1.798	1.77	1.6685	1.67	1.81	1.85	1.79	1.78	1.67	1.63

Table 4 (continued)

k	0.10	0.20	0.25	0.30	0.40	0.50	0.10	0.20	0.25	0.30	0.40	0.50	0.10	0.20	0.25	0.30	0.40	0.50
λ	0.40	0.40	0.40	0.40	0.40	0.40	0.50	0.50	0.50	0.50	0.50	0.50	0.10	0.10	0.10	0.10	0.10	0.10
γ	2.50	2.50	2.50	2.50	2.50	2.50	2.50	2.50	2.50	2.50	2.50	2.50	3.00	3.00	3.00	3.00	3.00	3.00
0.00	399.7	405	403	395	405	401	404	405.0	401.9	405	401	395	397	399	397	404	398.2	400
0.25	69.7	67.1	70.1	70.3	75.9	79.2	70.5	66.8	67.6	68.6	75.8	78.0	67.8	66.0	67.7	71.8	78.3	80.3
0.50	25.1	25.5	25.5	25.4	26.4	27.3	25.0	25.4	25.3	25.5	26.2	27.1	25.2	25.5	25.2	25.7	26.8	27.6
0.75	14.42	14.7	14.3	14.2	14.1	14.2	14.4	14.64	14.46	14.2	14.0	14.1	14.5	14.5	14.2	14.1	13.98	14.2
1.00	9.81	10.06	9.68	9.69	9.36	9.27	9.74	10.07	9.93	9.68	9.30	9.19	10.01	9.92	9.73	9.53	9.24	9.23
1.25	7.40	7.55	7.29	7.24	7.00	6.85	7.33	7.55	7.42	7.23	6.92	6.73	7.47	7.45	7.33	7.12	6.80	6.77
1.50	5.87	5.99	5.79	5.79	5.53	5.40	5.84	6.01	5.91	5.83	5.51	5.34	5.96	5.92	5.81	5.67	5.39	5.33
1.75	4.84	4.94	4.79	4.80	4.59	4.45	4.81	4.94	4.91	4.78	4.56	4.41	4.94	4.91	4.81	4.68	4.42	4.37
2.00	4.11	4.22	4.08	4.06	3.90	3.79	4.10	4.21	4.17	4.08	3.86	3.76	4.21	4.20	4.11	3.97	3.76	3.71
2.25	3.58	3.65	3.53	3.54	3.38	3.27	3.56	3.68	3.63	3.55	3.36	3.26	3.65	3.63	3.53	3.44	3.25	3.17
2.50	3.16	3.23	3.11	3.11	2.97	2.89	3.16	3.25	3.21	3.14	2.98	2.86	3.21	3.19	3.11	3.01	2.82	2.76
2.75	2.82	2.90	2.80	2.79	2.65	2.58	2.83	2.91	2.87	2.82	2.67	2.56	2.85	2.83	2.75	2.68	2.49	2.43
3.00	2.56	2.63	2.52	2.52	2.41	2.32	2.56	2.64	2.60	2.55	2.42	2.32	2.53	2.52	2.46	2.38	2.21	2.17
3.25	2.32	2.39	2.29	2.31	2.19	2.12	2.34	2.41	2.38	2.33	2.20	2.12	2.30	2.28	2.21	2.15	1.99	1.95
3.50	2.14	2.20	2.11	2.11	2.01	1.95	2.16	2.22	2.20	2.15	2.04	1.96	2.09	2.07	2.01	1.95	1.80	1.77
3.75	1.98	2.03	1.95	1.95	1.85	1.79	2.00	2.06	2.04	1.99	1.88	1.82	1.91	1.90	1.84	1.78	1.64	1.62
4.00	1.8262	1.89	1.81	1.81	1.72	1.66	1.86	1.9229	1.8997	1.85	1.74	1.68	1.75	1.75	1.69	1.63	1.505	1.49

k	0.10	0.20	0.25	0.30	0.40	0.50	0.10	0.20	0.25	0.30	0.40	0.50	0.10	0.20	0.25	0.30	0.40	0.50
λ	0.20	0.20	0.20	0.20	0.20	0.20	0.30	0.30	0.30	0.30	0.30	0.30	0.40	0.40	0.40	0.40	0.40	0.40
γ	3.00	3.00	3.00	3.00	3.00	3.00	3.00	3.00	3.00	3.00	3.00	3.00	3.00	3.00	3.00	3.00	3.00	3.00
0.00	401	399	405.8	401	406	398	395.4	399	398	397	400	405	399.6	407	396	402	408	400
0.25	67.2	67.5	65.5	72.2	79.5	80.5	67.4	66.0	69.1	71.1	78.7	82.2	68.4	66.9	67.6	71.3	79.2	81.9
0.50	25.0	25.2	25.2	25.4	26.8	27.6	25.1	25.1	25.3	25.4	26.5	28.1	25.2	25.4	25.1	25.5	27.1	27.9
0.75	14.4	14.4	14.50	14.1	14.0	14.2	14.42	14.4	14.2	13.9	13.9	14.3	14.48	14.5	14.3	14.0	14.0	14.2
1.00	9.91	9.93	9.93	9.49	9.21	9.27	9.88	9.89	9.73	9.42	9.20	9.27	9.84	9.98	9.64	9.45	9.19	9.14
1.25	7.48	7.47	7.46	7.11	6.79	6.72	7.42	7.45	7.32	7.06	6.78	6.78	7.43	7.44	7.26	7.13	6.74	6.71
1.50	5.95	5.96	5.96	5.67	5.40	5.29	5.96	5.95	5.81	5.61	5.35	5.29	5.93	5.94	5.76	5.62	5.37	5.26
1.75	4.93	4.90	4.93	4.66	4.43	4.32	4.90	4.90	4.82	4.67	4.43	4.36	4.91	4.94	4.80	4.65	4.40	4.33
2.00	4.18	4.20	4.21	3.99	3.75	3.67	4.16	4.19	4.08	3.95	3.73	3.68	4.16	4.19	4.08	3.95	3.73	3.63
2.25	3.64	3.63	3.65	3.44	3.23	3.15	3.63	3.62	3.54	3.42	3.21	3.17	3.62	3.65	3.55	3.44	3.23	3.14
2.50	3.20	3.21	3.20	3.01	2.82	2.74	3.20	3.20	3.13	3.02	2.82	2.74	3.18	3.22	3.13	3.03	2.84	2.75
2.75	2.85	2.85	2.85	2.67	2.49	2.42	2.85	2.85	2.77	2.67	2.50	2.44	2.86	2.87	2.80	2.70	2.51	2.43
3.00	2.55	2.56	2.55	2.38	2.23	2.16	2.56	2.56	2.50	2.39	2.23	2.18	2.57	2.59	2.51	2.42	2.26	2.19
3.25	2.31	2.30	2.31	2.15	2.01	1.95	2.32	2.32	2.25	2.16	2.00	1.97	2.35	2.35	2.27	2.20	2.03	1.98
3.50	2.10	2.11	2.10	1.95	1.82	1.76	2.11	2.11	2.05	1.98	1.83	1.79	2.14	2.16	2.09	2.01	1.85	1.80
3.75	1.92	1.92	1.92	1.78	1.66	1.62	1.94	1.93	1.88	1.79	1.67	1.64	1.98	1.98	1.91	1.84	1.71	1.65
4.00	1.77	1.77	1.77	1.64	1.52	1.49	1.78	1.79	1.74	1.66	1.54	1.51	1.81	1.83	1.76	1.70	1.57	1.52

Table 4 (continued)

k	0.10	0.20	0.25	0.30	0.40	0.50	0.10	0.20	0.25	0.30	0.40	0.50	0.10	0.20	0.25	0.30	0.40	0.50
λ	0.50	0.50	0.50	0.50	0.50	0.50	0.10	0.10	0.10	0.10	0.10	0.10	0.20	0.20	0.20	0.20	0.20	0.20
γ	3.00	3.00	3.00	3.00	3.00	3.00	4.00	4.00	4.00	4.00	4.00	4.00	4.00	4.00	4.00	4.00	4.00	4.00
0.00	402	398.6	403	406	404	400	405	402	403	400	398	401	400	402	396.1	401	396	405
0.25	68.1	66.7	67.9	71.7	78.9	83.0	66.3	65.4	68.8	73.5	80.2	77.4	66.4	66.0	69.0	73.7	81.8	79.8
0.50	25.1	25.3	25.2	25.6	26.8	27.7	25.3	25.0	25.2	25.8	27.5	27.7	25.5	25.1	25.1	25.7	27.2	28.0
0.75	14.5	14.53	14.2	14.1	13.8	14.2	14.7	14.3	14.1	13.9	14.1	14.4	14.7	14.3	13.94	13.9	14.1	14.4
1.00	9.83	9.93	9.65	9.45	9.12	9.13	10.07	9.76	9.48	9.24	9.10	9.31	10.08	9.75	9.35	9.19	9.09	9.31
1.25	7.40	7.39	7.21	7.09	6.72	6.61	7.64	7.35	7.11	6.85	6.68	6.84	7.63	7.31	7.02	6.84	6.66	6.80
1.50	5.88	5.91	5.73	5.61	5.33	5.22	6.15	5.92	5.66	5.44	5.26	5.39	6.09	5.88	5.60	5.44	5.19	5.36
1.75	4.87	4.90	4.79	4.64	4.37	4.29	5.12	4.90	4.69	4.50	4.34	4.41	5.08	4.86	4.65	4.47	4.30	4.39
2.00	4.14	4.17	4.08	3.94	3.71	3.61	4.38	4.21	4.02	3.82	3.64	3.71	4.38	4.16	3.97	3.80	3.63	3.69
2.25	3.61	3.63	3.55	3.45	3.20	3.13	3.83	3.66	3.47	3.29	3.13	3.20	3.81	3.64	3.43	3.28	3.12	3.16
2.50	3.20	3.22	3.13	3.04	2.82	2.75	3.38	3.19	3.04	2.88	2.74	2.78	3.39	3.18	2.99	2.86	2.71	2.75
2.75	2.86	2.88	2.80	2.72	2.51	2.45	3.01	2.82	2.67	2.53	2.40	2.44	3.01	2.81	2.63	2.52	2.37	2.41
3.00	2.59	2.60	2.54	2.45	2.27	2.21	2.69	2.51	2.35	2.23	2.12	2.16	2.68	2.51	2.34	2.23	2.10	2.14
3.25	2.36	2.37	2.31	2.23	2.07	2.01	2.39	2.24	2.09	1.98	1.88	1.92	2.41	2.23	2.08	1.98	1.87	1.90
3.50	2.16	2.18	2.12	2.05	1.90	1.84	2.17	2.00	1.88	1.78	1.68	1.71	2.17	1.99	1.87	1.78	1.68	1.71
3.75	2.01	2.01	1.96	1.89	1.75	1.70	1.96	1.81	1.68	1.59	1.52	1.54	1.96	1.81	1.68	1.59	1.52	1.54
4.00	1.86	1.86	1.82	1.75	1.61	1.56	1.77	1.64	1.53	1.44	1.38	1.40	1.78	1.64	1.51	1.45	1.38	1.40

k	0.10	0.20	0.25	0.30	0.40	0.50	0.10	0.20	0.25	0.30	0.40	0.50	0.10	0.20	0.25	0.30	0.40	0.50
λ	0.30	0.30	0.30	0.30	0.30	0.30	0.40	0.40	0.40	0.40	0.40	0.40	0.50	0.50	0.50	0.50	0.50	0.50
γ	4.00	4.00	4.00	4.00	4.00	4.00	4.00	4.00	4.00	4.00	4.00	4.00	4.00	4.00	4.00	4.00	4.00	4.00
0.00	396	395	403	397	400	400	403	397	396	400	403	399	404	403	405	406	396.9	405
0.25	65.4	65.2	69.7	73.6	83.8	82.2	66.9	65.3	68.7	73.9	83.5	82.8	67.4	65.4	70.3	74.4	81.7	83.7
0.50	25.3	25.1	25.1	25.9	27.9	28.0	25.4	25.0	24.8	25.7	28.1	28.7	25.3	24.9	25.2	25.9	27.6	28.7
0.75	14.6	14.3	14.0	14.0	14.1	14.4	14.7	14.2	13.9	13.8	14.1	14.3	14.6	14.1	14.0	13.8	13.92	14.3
1.00	10.01	9.69	9.33	9.26	9.06	9.27	10.02	9.67	9.29	9.17	9.04	9.15	9.92	9.68	9.38	9.09	8.95	9.12
1.25	7.57	7.29	7.07	6.78	6.62	6.75	7.54	7.26	6.93	6.78	6.59	6.67	7.45	7.19	6.99	6.71	6.52	6.62
1.50	6.07	5.83	5.61	5.43	5.20	5.28	6.05	5.82	5.53	5.36	5.17	5.23	5.97	5.76	5.57	5.30	5.11	5.12
1.75	5.06	4.83	4.62	4.45	4.28	4.33	5.03	4.83	4.57	4.42	4.25	4.26	4.95	4.78	4.61	4.39	4.19	4.24
2.00	4.33	4.13	3.95	3.78	3.62	3.64	4.33	4.13	3.89	3.75	3.58	3.61	4.24	4.09	3.91	3.74	3.53	3.59
2.25	3.77	3.60	3.42	3.26	3.08	3.13	3.77	3.59	3.37	3.25	3.09	3.11	3.70	3.55	3.39	3.23	3.06	3.10
2.50	3.33	3.14	2.99	2.85	2.70	2.71	3.34	3.14	2.95	2.84	2.69	2.69	3.30	3.14	2.99	2.82	2.68	2.71
2.75	2.97	2.79	2.64	2.49	2.37	2.39	2.98	2.80	2.63	2.49	2.37	2.38	2.95	2.79	2.66	2.51	2.36	2.38
3.00	2.66	2.49	2.34	2.21	2.10	2.12	2.68	2.50	2.33	2.22	2.10	2.11	2.65	2.51	2.40	2.24	2.12	2.12
3.25	2.39	2.23	2.10	1.98	1.86	1.88	2.41	2.25	2.09	1.98	1.88	1.89	2.40	2.27	2.14	2.02	1.91	1.91
3.50	2.16	2.00	1.88	1.77	1.67	1.70	2.18	2.04	1.89	1.79	1.69	1.71	2.19	2.06	1.96	1.83	1.72	1.74
3.75	1.96	1.81	1.70	1.61	1.52	1.53	2.00	1.85	1.70	1.62	1.54	1.55	2.01	1.88	1.78	1.66	1.57	1.58
4.00	1.79	1.65	1.54	1.45	1.39	1.40	1.82	1.68	1.55	1.47	1.41	1.41	1.84	1.73	1.63	1.52	1.437	1.46

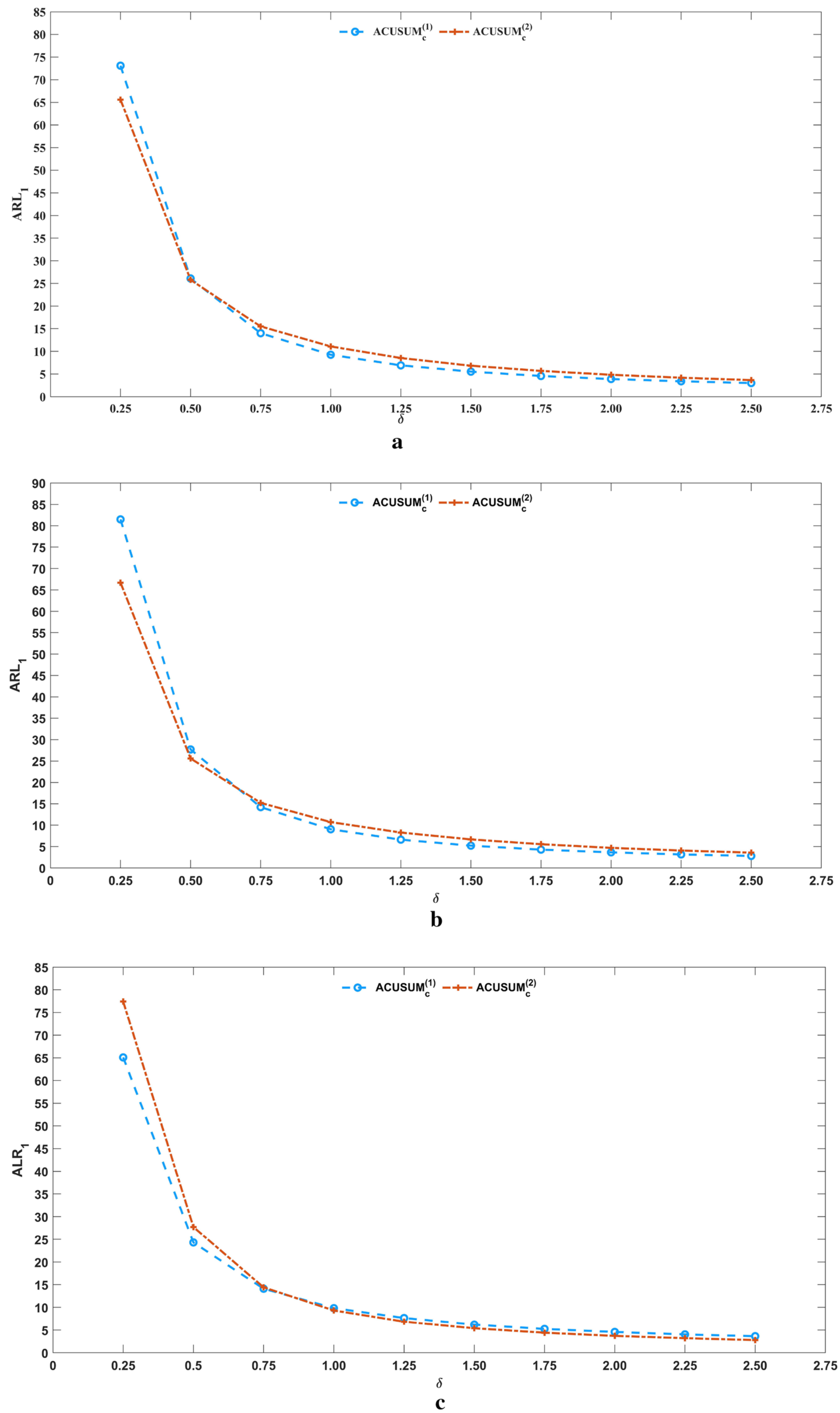
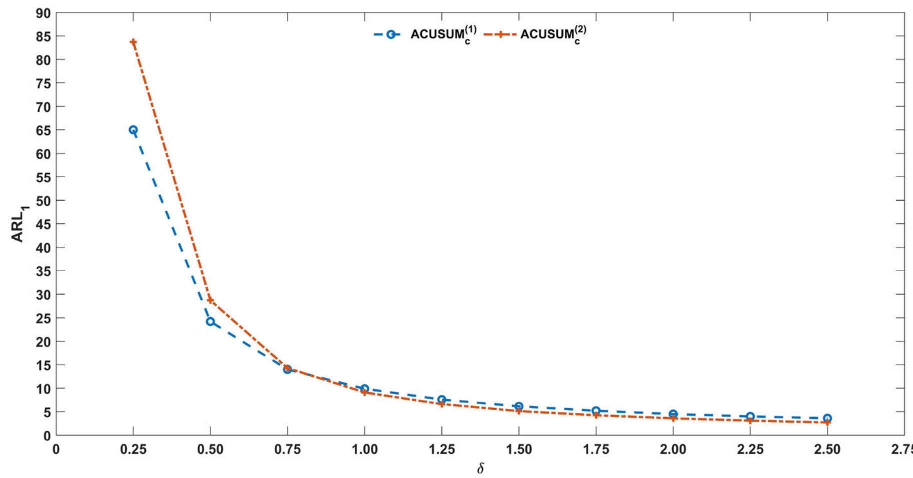


Fig. 2 **a** ARL comparison between proposed ACUSUM$_c^{(1)}$ and ACUSUM$_c^{(2)}$ control charts when $\lambda = 0.10$, $\gamma = 1.00$, $k = 0.50$, and$\delta^+_{\min} = 0.50$. **b** ARL comparison between proposed ACUSUM$_c^{(1)}$ and ACUSUM$_c^{(2)}$ control charts when $\lambda = 0.50$, $\gamma = 1.00$, $k = 0.50$, and$\delta^+_{\min} = 0.50$. **c** ARL comparison between proposed ACUSUM$_c^{(1)}$ and ACUSUM$_c^{(2)}$ control charts when $\lambda = 0.10$, $\gamma = 4.00$, $k = 0.50$, and$\delta^+_{\min} = 0.50$. **d** ARL comparison between proposed ACUSUM$_c^{(1)}$ and ACUSUM$_c^{(2)}$ control charts when $\lambda = 0.50$, $\gamma = 4.00$, $k = 0.50$, and$\delta^+_{\min} = 0.50$

Fig. 2 continued



d

control charts. The comparison is based on numerical results and visual presentations. More details are given as follows:

(a) At small values of $\gamma = 1$ and $\lambda = 0.10$ along $k = 0.5$, the proposed ACUSUM$_c^{(2)}$ control chart outperforms for small shifts (i.e., $\delta \leq 0.5$) as compared to the proposed ACUSUM$_c^{(1)}$ control chart (see Tables 3, 4 and Fig. 2a). It means, the proposed ACUSUM$_c^{(1)}$ control chart works better for moderate-to-large shift (i.e., $\delta > 0.5$).

(b) As $\lambda > 0.10$ (e.g., $\lambda = 0.50$), the proposed ACUSUM$_c^{(2)}$ control chart identify earlier signal relative to the proposed ACUSUM$_c^{(1)}$ control chart which can be seen in Tables 3, 4 and Fig. 2b as well.

(c) At large values of γ and k such that $\gamma = 5$ and $k = 0.5$ along $\lambda = 0.10$, the proposed ACUSUM$_c^{(1)}$ control chart shows good diagnose ability for small shifts (i.e., $\delta \leq 0.75$) against the proposed ACUSUM$_c^{(2)}$ control chart (see Tables 3, 4 and Fig. 2c). It implies that proposed ACUSUM$_c^{(2)}$ control chart works well for moderate-to-large shift (i.e., $\delta > 0.75$).

(d) Like proposed ACUSUM$_c^{(2)}$ control chart in point (b), as $\lambda > 0.10$ (e.g., $\lambda = 0.50$), the proposed ACUSUM$_c^{(1)}$ control chart continues with superiority as mentioned in point (c) and this point can be seen in Fig. 2d and Tables 3, 4.

(e) It can be concluded from points (a)-(d), at the different parameters' combinations, the proposed ACUSUM$_c^{(1)}$ and ACUSUM$_c^{(2)}$ control charts offer better detection ability for different sizes of shift.

5.2 Proposed and Other Control Charts

The proposed ACUSUM$_c^{(1)}$ and ACUSUM$_c^{(2)}$ control charts performance are compared against some existing control charts. The performance of the control charts is assessed at a single shift and at a certain range of shifts as well to convey adaptive idea. For the said objective, the classical CUSUM [2], ACUSUM [9, 11], and ACUSUM$_E$ [7] control charts are considered.

5.2.1 Proposed Versus Classical CUSUM Control Chart

The proposed ACUSUM$_c^{(1)}$ and ACUSUM$_c^{(2)}$ control charts keep smaller ARL$_1$ values against the classical CUSUM control chart when different values of parameters are considered. For instance, at $k = 0.5$, the ARL$_1$ values of the classical CUSUM control chart are 85.87 and 28.49 for $\delta = 0.25 \text{and} 0.5$, respectively, while the proposed ACUSUM$_c^{(1)}$ control chart at any combination of k, λ, and γ values has smaller ARL$_1$ values for the same shifts (see Table 5 and Fig. 3a). Likewise, the proposed ACUSUM$_c^{(2)}$ control chart also shows dominance as compared to the classical CUSUM control chart (see Table 5 and Fig. 3a). It can be concluded that the proposed ACUSUM$_c^{(1)}$ and ACUSUM$_c^{(2)}$ control charts outperform versus the classical CUSUM control chart when shift lies between $0.25 < \delta \leq 0.75$ intervals. Similarly, as $k > 0.5$ increases, the proposed ACUSUM$_c^{(1)}$ and ACUSUM$_c^{(2)}$ control charts persist with outstanding detection ability relative to the classical CUSUM control chart (see Table 5 and Fig. 3b).

The proposed ACUSUM$_c^{(1)}$ and ACUSUM$_c^{(2)}$ control charts also show edge in terms of overall performance against

Table 5 ARL values of $ACUSUM_c^{(1)}$, $ACUSUM_c^{(2)}$ $ACUSUM_E$, ACUSUM, and classical CUSUM control charts when $ARL_0 = 400$

	$\delta^+_{min} = 0.50$						$\delta^+_{min} = 1.00$				
	$ACUSUM_c^{(1)}$	$ACUSUM_c^{(2)}$	ACUSUM	OCUSUM	Classical CUSUM		$ACUSUM_c^{(1)}$	OCUSUM	ACUSUM	Classical CUSUM	
k	0.50	0.50		0.25	0.25	0.5 0	1.00	0.50		1.00	1.50
λ	0.20	0.10					0.10				
δ	$h_{ACUSUM_c^{(1)}} =$ 3.43	$h_{ACUSUM_c^{(2)}} =$ 5.13	$h_{ACUSUM} =$ 1.115	5.96	$h = 6.86\ 6$	$h =$ 4.173	$h_{ACUSUM_c^{(1)}} =$ 4.18	$L = 3.8755$	$h_{ACUSUM} =$ 1.0355	$h =$ 2.556	h = 1.387
0.00	400	400	400	400	399.69	399.87	400	400	399.37	397	400.1
0.25	65.2	65.6	63.86	57.70	64.34	85.87	84.3	80.55	85.75	210	168.68
0.50	24.2	25.8	24.03	21.70	24.27	28.49	28.0	26.49	28.45	72.5	74.87
0.75	14.2	11.08	13.82	12.57	14.1	13.93	14.0	13.13	13.88	29.7	35.52
1.00	9.83	6.83	9.53	8.77	9.88	8.73	8.84	8.20	8.67	14.8	18.31
1.50	6.21	4.84	5.78	5.49	6.20	4.92	4.92	4.63	4.84	6.51	6.5
2.00	4.57	3.66	4.12	4.06	4.56	3.46	3.46	3.26	3.36	4.30	3.29
2.50	3.65	2.94	3.20	3.24	3.65	2.7	2.71	2.56	2.59	3.37	2.13
3.00	3.07	2.46	2.62	2.72	3.06	2.26	2.26	2.15	2.12	2.87	1.59
3.50	2.64	2.15	2.24	2.36	2.65	1.98	1.97	1.87	1.80	2.57	1.30
4.00	2.33	1.78	1.98	2.13	2.33	1.77	1.77	1.66	1.55	2.33	1.14
4.50	1.98	1.45	1.72	1.99	2.23	1.39	1.45	1.45	1.23	2.18	1.05
5.00	1.33	1.23	1.58	1.89	2.01	1.37	1.23	1.26	1.18	2.07	1.02

Table 5 (continued)

	IACUSUM									
	$\delta^{+}_{\min} = 0.50$					$\delta^{+}_{\min} = 1.00$				
λ	0.10	0.20	0.30	0.40	0.50	0.10	0.20	0.30	0.40	0.50
h	3.563	4.357	5.23	6.02	6.674	4.1795	4.185	4.341	4.6418	4.967
0.00	400	400	400	401	400	401	402	401	402	401
0.25	56.73	59.09	60.91	61.88	63.41	84.43	84.06	84.54	86.31	87
0.50	19.00	20.77	22.51	23.71	24.6	27.35	27.14	27.32	27.89	28.3
0.75	10.03	11.35	12.61	13.49	14.19	12.9	12.85	12.9	13.23	13.61
1.00	6.45	7.37	8.37	9.12	9.65	7.83	7.73	7.81	8.06	8.37
1.25	4.62	5.34	6.07	6.67	7.11	5.42	5.35	5.44	5.64	5.85
1.50	3.56	4.09	4.66	5.14	5.51	4.08	4.03	4.11	4.27	4.44
1.75	2.86	3.28	3.72	4.11	4.42	3.22	3.20	3.26	3.4	3.55
2.00	2.37	2.70	3.08	3.39	3.63	2.66	2.64	2.69	2.81	2.93
2.25	2.03	2.30	2.58	2.85	3.07	2.25	2.24	2.29	2.38	2.49
2.50	1.77	2.00	2.23	2.46	2.63	1.94	1.94	1.99	2.06	2.15
2.75	1.57	1.76	1.96	2.15	2.30	1.72	1.71	1.75	1.82	1.89
3.00	1.42	1.57	1.74	1.90	2.04	1.54	1.54	1.57	1.62	1.69
3.25	1.31	1.43	1.57	1.71	1.82	1.40	1.40	1.42	1.46	1.52
3.50	1.21	1.31	1.43	1.55	1.65	1.29	1.29	1.31	1.35	1.39
3.75	1.14	1.22	1.32	1.42	1.50	1.20	1.20	1.22	1.25	1.29
4.00	1.09	1.15	1.23	1.31	1.38	1.14	1.14	1.15	1.17	1.2

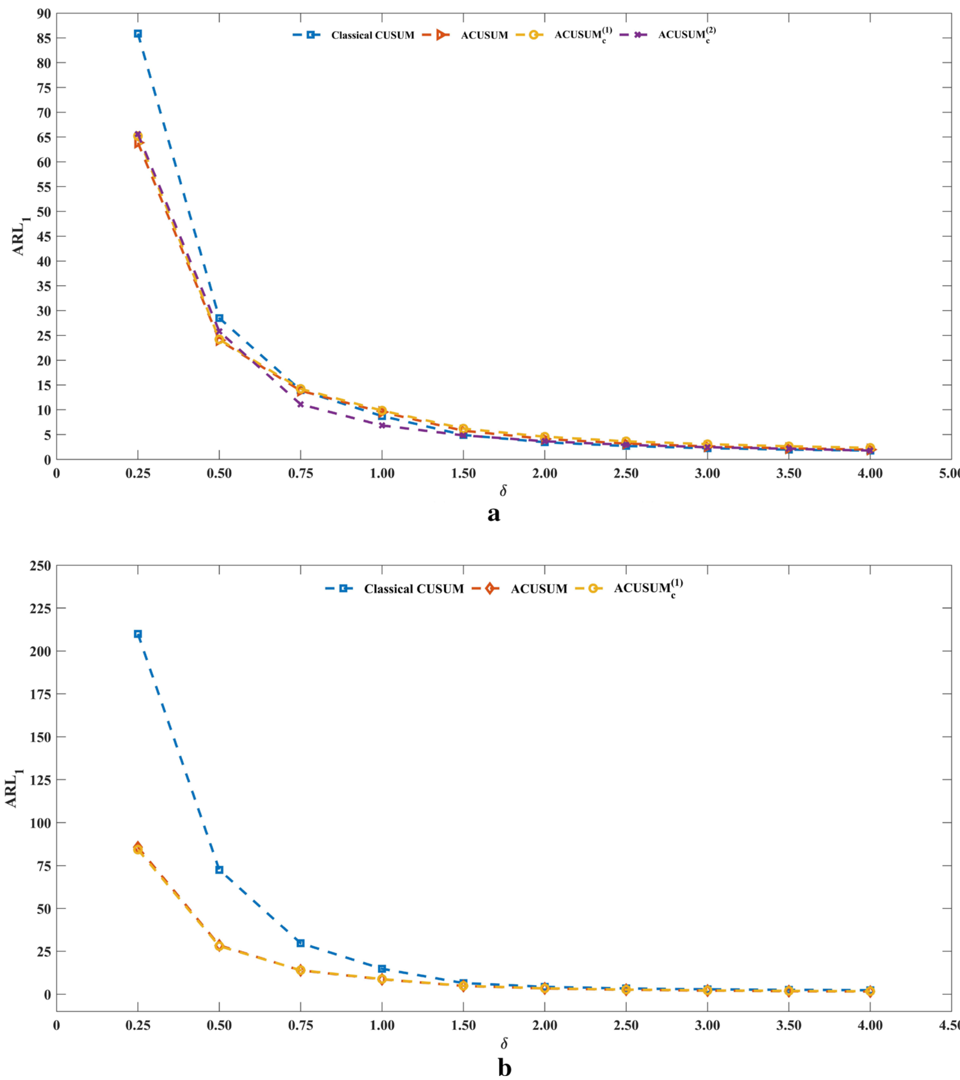


Fig. 3 **a** ARL comparison among ACUSUM$_c^{(1)}$($\gamma = 4.00$), ACUSUM$_c^{(2)}$($\gamma = 4.00$), classical CUSUM ($k = 0.50$), and ACUSUM control charts when $\lambda = 0.20$and$\delta_{\min}^{+} = 0.50$. **b** ARL comparison among ACUSUM$_c^{(1)}$($\gamma = 4.00$), ACUSUM$_c^{(2)}$($\gamma = 4.00$), classical CUSUM ($k = 1$), and ACUSUM control charts when $\lambda = 0.10$and$\delta_{\min}^{+} = 1.00$

the classical CUSUM control chart. For example, the classical CUSUM control chart has lower EQL, PCI, and RARL values as compared to the proposed ACUSUM$_c^{(1)}$ control chart, but the proposed ACUSUM$_c^{(2)}$ control chart shows superiority. For instance, the EQL, PCI, and RARL values of the ACUSUM$_c^{(2)}$ control chart are 8.77, 1.00, and 1.00 (see Table 6). From the findings, it can be concluded that the proposed control charts somehow perform better for a single shift and in terms of overall assessment as well.

5.2.2 Proposed Versus ACUSUM Control Chart

The ACUSUM control chart is proposed by Sparks [9]. The ACUSUM control chart is effective to detect different sizes of shift. In the comparison of the proposed ACUSUM$_c^{(2)}$ control chart, the ACUSUM control chart is less efficient at $\delta \geq 0.75$ to diagnose earlier shifts. For example, at $\delta = 0.75$ ($\delta_{\min}^{+} = 0.50$, $\lambda = 0.20$, and $\gamma = 1.00$), the ARL$_1$ 11.08 value is smaller than the ARL$_1$ value of the ACUSUM control chart. Similarly, at $\delta = 2.5\,0$, the ARL$_1$ 4.12 value which belongs to the ACUSUM control char is larger relative to the proposed ACUSUM$_c^{(2)}$ control chart ARL$_1$ value (see Table 5 and Fig. 3a). In contrary, the proposed ACUSUM$_c^{(1)}$ control chart performs better against the ACUSUM control chart only at $\delta = 0.50$ (see Table 5 and Fig. 3a); otherwise, it has inferior performance. Furthermore, when $\delta_{\min}^{+} = 1.00$, the proposed ACUSUM$_c^{(1)}$ and ACUSUM control charts perform almost equally. It means that the ACUSUM$_c^{(2)}$ control chart shows outstanding performance against the ACUSUM control chart at the specific values of parameters.

In terms of comprehensive performance, the ACUSUM control chart keeps large values of the EQL, PCI, and RARL against the proposed ACUSUM$_c^{(1)}$ and ACUSUM$_c^{(2)}$ control charts. For instance, at $\delta_{\min}^{+} = 1.00$, the EQL, PCI, and RARL

Table 6 EQL, PCI, and RARL values of control charts

	$\delta^+_{\min}$	λ	γ		EQL	PCI	RARL
$AEWMA_E$		0.0398	2.899		14.17	1.33	1.33
$ACUSUM_E$	0.50	0.0398	2.899		13.28	1.25	1.23
$ACUSUM_c^{(1)}$	0.50	0.0398	2.899	3.62	11.56	1.09	1.05
$ACUSUM_c^{(2)}$	0.50	0.0398	2.899	3.63	10.62	1.00	1.00
$AEWMA_E$		0.20	2.50		15.84	1.48	1.66
$ACUSUM_E$	0.50	0.20	2.50		12.56	1.18	1.17
$ACUSUM_c^{(1)}$	0.50	0.20	2.50	3.57	11.48	1.08	1.04
$ACUSUM_c^{(2)}$	0.50	0.20	2.50	3.93	10.68	1.00	1.00
$AEWMA_E$		0.1253	2.7765		12.70	1.26	1.29
$ACUSUM_E$	1.00	0.1253	2.7765		11.74	1.17	1.18
$ACUSUM_c^{(1)}$	1.00	0.1253	2.7765		10.07	1.00	1.00
$ACUSUM_c^{(2)}$	1.00	0.1253	2.7765	4.40	10.08	1.00	1.00
$AEWMA_E$		0.30	2.50		17.32	1.72	1.88
$ACUSUM_E$	1.00	0.30	2.50		11.72	1.16	1.19
$ACUSUM_c^{(1)}$	1.00	0.30	2.50		10.07	1.00	1.00
$ACUSUM_c^{(2)}$	1.00	0.30	2.50		10.09	1.00	1.00
Classical CUSUM					9.53	1.09	1.13
ACUSUM	0.50				10.37	1.18	1.18
$ACUSUM_c^{(1)}$	0.50	0.20	4.00		11.01	1.25	1.24
$ACUSUM_c^{(2)}$	0.50	0.10	1.00		8.77	1.00	1.00
ACUSUM	1.00				15.73	1.67	1.78
$ACUSUM_c^{(1)}$	1.00	0.100	4.00		9.40	1.00	1.00
$ACUSUM_c^{(2)}$	1.00	0.100	1.00		9.54	1.01	1.01
OCUSUM							
$ACUSUM_c^{(1)}$							
$ACUSUM_c^{(2)}$							

values of the ACUSUM control charts are 15.73, 1.67, and 1.78, respectively (see Table 5). The same analysis is also true at $\delta^+_{\min} = 0.50$ but only for the proposed $ACUSUM_c^{(2)}$ control chart (see Table 6). In brief, the $ACUSUM_c^{(2)}$ control chart is a better choice in terms of overall performance as compared to the proposed $ACUSUM_c^{(1)}$ control chart against the ACUSUM control chart.

5.2.3 Proposed Versus $AEWMA_E$ Control Chart

The analysis at range of $\delta \in [0.5, 4]$ reveals the ARL_1 values of the proposed $ACUSUM_c^{(1)}$ and $ACUSUM_c^{(2)}$ control charts are smaller as compared to the $AEWMA_E$ control chart at $\delta^+_{\min} = 0.5 \text{and} 1.00$ along different values of λ and γ for the specific range of shift (i.e., $0 < \delta \leq 1.5$). For instance, at $\delta = 0.25$ ($\delta^+_{\min} = 0.5$, $\lambda = 0.0398$, and $\gamma = 2.889$), the 71.20 and 89.84 are ARL_1 values of the proposed $ACUSUM_c^{(1)}$ and $ACUSUM_c^{(2)}$ control charts, respectively, whereas the $AEWMA_E$ control chart ARL_1 value is 115.30. It shows that the $AEWMA_E$ control chart has less detection ability against the proposed $ACUSUM_c^{(1)}$ and $ACUSUM_c^{(2)}$ control charts (see Table 7 and Fig. 4a). Similarly, at $\delta = 1.5$, the proposed $ACUSUM_c^{(1)}$ and $ACUSUM_c^{(2)}$ control charts also show superiority versus the $AEWMA_E$ control chart. Furthermore, as λ increases, the $AEWMA_E$ control chart ARL_1 increases as well relative to the proposed $ACUSUM_c^{(1)}$ and $ACUSUM_c^{(2)}$ control charts. For example, at $\lambda = 0.20$ and $\delta = 0.25$ ($\delta^+_{\min} = 0.50$ and $\gamma = 2.50$), the 72.46 and 87.39 are ARL_1 values of the proposed $ACUSUM_c^{(1)}$ and $ACUSUM_c^{(2)}$ control charts, respectively, although the $AEWMA_E$ control chart ARL_1 value is larger (see Table 3 and Fig. 4b). Correspondingly, at $\delta = 0.50$, the proposed $ACUSUM_c^{(1)}$ and $ACUSUM_c^{(2)}$ control charts perform better, too. Likewise,

Table 7 Comparison of zero-state ARL values among $ACUSUM_c$, $ACUSUM_E$, and $AEWMA_E$ control charts when $ARL_0 = 500$

	Parameters				δ										
	δ^+_{min}	λ	γ		0.25	0.50	0.75	1.00	1.50	2.00	2.50	3.00	3.50	4.00	5.00
Range [0.5, 4]															
$AEWMA_E$		0.0398	2.899		115.30	36.71	20.08	13.53	7.77	4.96	3.25	2.19	1.58	1.26	1.04
$ACUSUM_E$	0.50	0.0398	2.899		96.34	31.47	17.66	12.18	7.40	5.15	3.75	2.79	2.12	1.67	1.19
$ACUSUM_c^{(1)}$	0.50	0.0398	2.899	3.62	71.20	25.81	14.86	10.38	6.49	4.78	3.81	3.20	2.77	2.43	2.00
$ACUSUM_c^{(2)}$	0.50	0.0398	2.899	3.63	89.48	29.36	15.03	9.77	5.65	3.91	2.95	2.32	1.90	1.59	1.13
$AEWMA_E$		0.20	2.50		260.79	79.05	29.31	14.68	6.44	3.87	2.63	1.91	1.47	1.23	1.03
$ACUSUM_E$	0.50	0.20	2.50		95.54	31.98	17.76	11.90	6.84	4.58	3.30	2.49	1.95	1.59	1.17
	0.50	0.20	2.50	3.57	72.46	25.79	14.80	10.36	6.43	4.72	3.75	3.13	2.69	2.34	1.83
$ACUSUM_c^{(2)}$	0.50	0.20	2.50	3.93	87.39	29.02	14.98	9.75	5.72	4.00	3.04	2.45	2.03	1.73	1.29
Range [1, 4]															
$AEWMA_E$		0.1253	2.7765		168.55	45.02	19.64	11.69	6.17	4.00	2.79	2.03	1.55	1.26	1.04
$ACUSUM_E$	1.00	0.1253	2.7765		147.49	39.25	17.42	10.57	5.81	3.99	3.00	2.37	1.91	1.57	1.17
$ACUSUM_c^{(1)}$	1.00	0.1253	2.7765	4.40	99.06	30.76	14.82	9.12	5.16	3.61	2.82	2.34	2.06	1.85	1.46
$ACUSUM_c^{(2)}$	1.00	0.1253	2.7765	4.40	100.23	31.19	14.81	9.23	5.12	3.60	2.82	2.34	2.05	1.85	1.46
$AEWMA_E$		0.30	2.50		276.61	93.60	35.27	16.66	6.52	3.75	2.54	1.87	1.46	1.23	1.03
$ACUSUM_E$	1.00	0.30	2.50		153.52	40.58	17.87	10.70	5.72	3.82	2.80	2.17	1.75	1.45	1.11
$ACUSUM_c^{(1)}$	1.00	0.30	2.50	4.40	99.03	30.79	14.64	9.21	5.15	3.61	2.81	2.34	2.05	1.85	1.47
$ACUSUM_c^{(2)}$	1.00	0.30	2.50	4.40	98.93	31.00	14.89	9.16	5.16	3.61	2.81	2.34	2.05	1.85	1.46

at $\lambda = 0.30$, the proposed $ACUSUM_c^{(1)}$ and $ACUSUM_c^{(2)}$ control charts persist excellent performance. Besides, when $\delta^+_{min} = 1.00$ the proposed $ACUSUM_c^{(1)}$ and $ACUSUM_c^{(2)}$ control charts give outstanding performance against the control chart (see Table 7). At the similar lines, the proposed $ACUSUM_c^{(1)}$ and $ACUSUM_c^{(2)}$ control charts have outperformed against the $AEWMA_E$ control chart when range of $\delta \in [0.5, 4]$ is considered (see Fig. 4a, b).

In terms of overall performance evaluation, the proposed $ACUSUM_c^{(1)}$ and $ACUSUM_c^{(2)}$ control charts keep lower EQL, PCI, and RARL values against the $AEWMA_E$ control chart when a certain range of shift (i.e., $0 \le \delta \le 2$) is used. For example, at $\delta^+_{min} = 0.50$and $\lambda = 0.0398$, the 11.56 and 10.62 are EQL values of the $ACUSUM_c^{(1)}$ and $ACUSUM_c^{(2)}$ control charts, respectively, which are smaller as compared to the $AEWMA_E$ control chart (see Table 6). Likewise, at $\delta^+_{min} = 1.00$and $\lambda = 0.30$, the $AEWMA_E$ control chart PCI and RARL values are larger in the comparison of proposed $ACUSUM_c^{(1)}$ and $ACUSUM_c^{(2)}$ control charts PCI and RARL values (see Table 7). Besides, at δ^+_{min}, λ, and γ parameters values as mentioned in Table 5, the $AEWMA_E$ control chart depicts the inferior performance, too.

5.2.4 Proposed Versus $ACUSUM_E$ Control Charts

The proposed $ACUSUM_c^{(1)}$ and $ACUSUM_c^{(2)}$ control charts have superior performance against the $ACUSUM_E$ control chart when $\delta^+_{min} = 0.50$and1.00 along different values of λ and γ at the specific range of shift (i.e., $0 < \delta \le 1.5$). For instance, at $\delta = 0.25$ ($\delta^+_{min} = 0.5$, $\lambda = 0.0398$, and $\gamma = 2.889$), the 96.34 is the ARL_1 values of the $ACUSUM_E$ control chart which is larger as compared to the proposed $ACUSUM_c^{(1)}$ and $ACUSUM_c^{(2)}$ control charts (see Table 7 and Fig. 4a). It shows that the $ACUSUM_E$ control chart has less detection ability against the proposed $ACUSUM_C^{(1)}$ and $ACUSUM_C^{(2)}$ control charts. Similarly, at $\delta = 1.50$, the proposed $ACUSUM_c^{(1)}$ and $ACUSUM_c^{(2)}$ control charts demonstration supremacy over the $ACUSUM_E$ control chart (see Tables 7 and Fig. 4a). Additionally, as λ increases, the $ACUSUM_E$ control chart ARL_1 increases as well over the proposed $ACUSUM_c^{(1)}$ and $ACUSUM_c^{(2)}$ control charts. For example, at $\lambda = 0.20$ and $\delta = 0.25$ ($\delta^+_{min} = 0.50$ and $\gamma = 2.50$), the 72.46 and 87.39 are ARL_1 values of the proposed $ACUSUM_c^{(1)}$ and $ACUSUM_c^{(2)}$ control charts, respectively, while the $ACUSUM_E$ control chart ARL_1(i.e., $ARL_1 = 95.54$) value is larger (Table 7 and Fig. 4b). Similarly, at

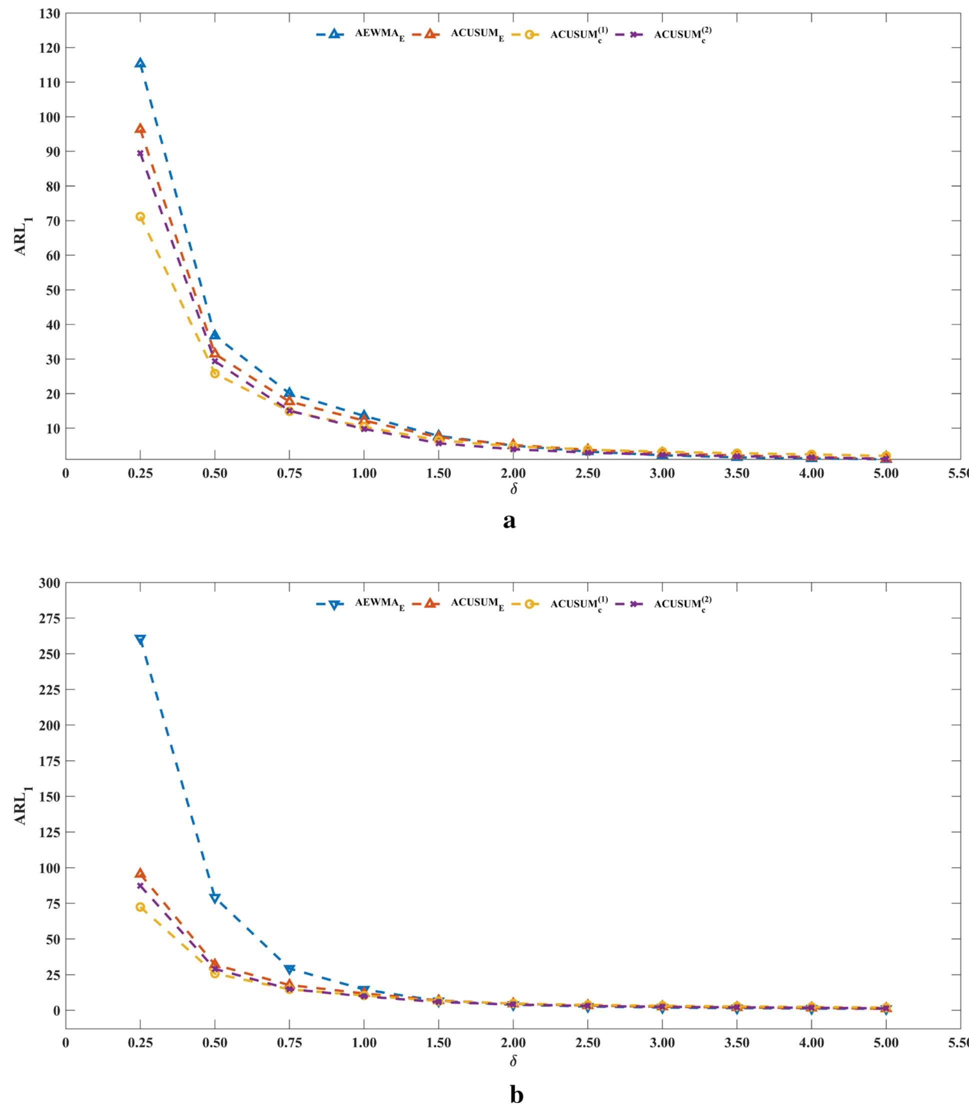


Fig. 4 **a** ARL comparison among ACUSUM$_c^{(1)}$, ACUSUM$_c^{(2)}$, AEWMA$_E$, and ACUSUM$_E$ control charts when $\lambda = 0.0398$, $\gamma = 2.899$, $k = 0.50$, and$\delta^+_{min} = 0.50$. **b** ARL comparison among ACUSUM$_c^{(1)}$, ACUSUM$_c^{(2)}$, AEWMA$_E$, and ACUSUM$_E$ control charts when $\lambda = 0.20$, $\gamma = 2.50$, $k = 0.50$, and$\delta^+_{min} = 0.50$

$\delta = 0.50$, the proposed ACUSUM$_c^{(1)}$ and ACUSUM$_c^{(2)}$ control charts perform better, too. Likewise, at $\lambda = 0.30$, the proposed ACUSUM$_c^{(1)}$ and ACUSUM$_c^{(2)}$ control charts persist superior performance. Besides, when $\delta^+_{min} = 1.00$, the proposed ACUSUM$_c^{(1)}$ and ACUSUM$_c^{(2)}$ control charts give outstanding performance against the ACUSUM$_E$ control chart, too (see Table 7 and Fig. 5a, b).

In terms of comprehensive assessment, the ACUSUM$_E$ control chart shows inferior performance against the proposed ACUSUM$_c^{(1)}$ and ACUSUM$_c^{(2)}$ control charts at a specific interval of shift (i.e., $0 \le \delta \le 2$). For example, at $\delta^+_{min} = 0.50$and $\lambda = 0.20$, the 11.48, 10.68, and 12.56 are EQL values of the ACUSUM$_c^{(1)}$, ACUSUM$_c^{(2)}$, ACUSUM$_E$ control charts, respectively. This analysis reveals that the ACUSUM$_E$ control chart has inferior performance (see Table 6). Furthermore, the analysis at δ^+_{min}, λ, and γ parameters values as mentioned in Table 6 also shows the superiority of the proposed ACUSUM$_c^{(1)}$ and ACUSUM$_c^{(2)}$ control charts.

5.2.5 Proposed Versus OCUSUM and IACUSUM Control Charts

The ARL values of the optimal CUSUM (OCUSUM) and IACUSUM control charts are taken from the study of Abbasi and Haq (2020) and provided in Table 5. The 80.55 and 26.49 are ARL$_1$ values of the OCUSUM control chart for $\delta = 0.25$and0.50, respectively, at $k = 0.50$, while the proposed ACUSUM$_c^{(1)}$(ARL$_1 = 73.1$and 26.1) and ACUSUM$_c^{(2)}$(ARL$_1 = 65.6$and 25.8) control charts have smaller ARL$_1$ for same shifts (see Tables 3, 4 versus 5). It shows that the proposed ACUSUM$_c^{(2)}$ performs

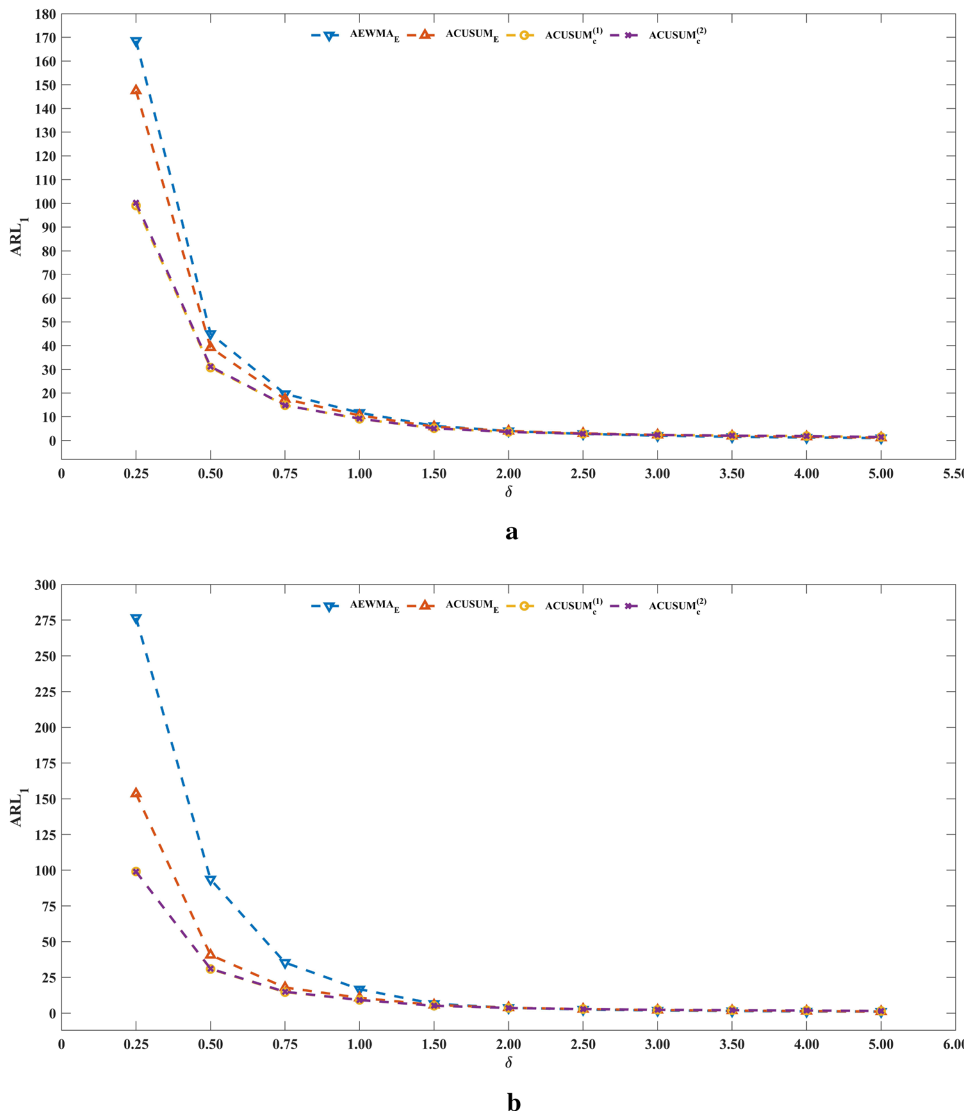


Fig. 5 **a** ARL comparison among $ACUSUM_c^{(1)}$, $ACUSUM_c^{(2)}$, $AEWMA_E$, and $ACUSUM_E$ control charts when $\lambda = 0.1253$, $\gamma = 2.7765$, $k = 0.50$, and$\delta^+_{\min} = 1.00$. **b** ARL comparison among $ACUSUM_c^{(1)}$, $ACUSUM_c^{(2)}$, $AEWMA_E$, and $ACUSUM_E$ control charts when $\lambda = 0.30$, $\gamma = 2.50$, $k = 1.00$, and$\delta^+_{\min} = 1.00$

better as compared to the proposed $ACUSUM_c^{(1)}$ against the OCUSUM control chart for small shifts. In contrary, δ increases (i.e., $\delta > 0.50$), the OCUSUM has dominance over the proposed control charts. Similarly, the IACUSUM control chart also shows efficient performance, too (see Table 5).

5.3 Recommendations When to Use $ACUSUM_c^{(1)}$ and $ACUSUM_c^{(2)}$ Control Charts

As mentioned in Sect. 4.2, the performance of the proposed control charts truly depends on the combinations of parameters (i.e., k, λ, and γ) and their values, but the above-mentioned points (a)–(e) in Sect. 5.1 will be helpful for practitioners, quality experts, researchers, and engineers to correctly identify the situation in real-life when to use the proposed $ACUSUM_c^{(1)}$ and $ACUSUM_c^{(2)}$ control charts to monitor the process location shift precisely. Furthermore, the comparative analysis of the proposed $ACUSUM_c^{(1)}$ and $ACUSUM_c^{(2)}$ control charts for a single shift and a broad range of shift given in Sect. 5.2 can be used as a benchmark to properly recognize the use of the proposed $ACUSUM_c^{(1)}$ and $ACUSUM_c^{(2)}$ control charts to solve real-life process problems for many situations at different choices of parameters against other (classical CUSUM, ACUSUM, $AEWMA_E$, $ACUSUM_E$, OCUSUM, and IACUSUM) control charts.

6 An Illustrate Example

This section demonstrates how the $ACUSUM_c^{(1)}$, $ACUSUM_E$, $AEWMA_E^{(1)}$ ($AEWMA_E$ based on Huber function) and the classical CUSUM control charts can employ with numerical data to show the implementation procedure for practical point of view. Also, a comparison is provided to show the efficiency of the proposed $ACUSUM_c^{(1)}$ control

Table 8 Example of $ACUSUM_E$, classical CUSUM, $AEWMA_E^{(1)}$, and proposed $ACUSUM_c^{(1)}$ control charts

	Real-life data		$ACUSUM_E$			Classical CUSUM			$ACUSUM_C^{(1)}$			$AEWMA_E^{(1)}$			
	x_i	z_i	Z_i^+		H_{ACUSUM_E}	C_i^+		H	$ACUSUM_i^{(1)+}$		$H_{ACUSUM_c^{(1)}}$	$AEWMA_{Ei}^{(1)}$			
			$\delta = 0.00$	1.50		0.00	1.50		0.00	1.50		0.00	1.50	LCL	UCL
1	9.45	− 0.55	0.00	0.00	4.39	0.00	0.00	2.21	0.00	0.00	4.17	− 0.17	− 0.17	− 0.89	0.89
2	7.99	− 2.01	0.00	0.00	4.39	0.00	0.00	2.21	0.00	0.00	4.17	− 0.72	− 0.72	− 1.08	1.08
3	9.29	− 0.71	0.00	0.00	4.39	0.00	0.00	2.21	0.00	0.00	4.17	− 0.72	− 0.72	− 1.17	1.17
4	11.66	1.66	1.16	1.16	4.39	0.66	0.66	2.21	1.16	1.16	4.17	0.00	0.00	− 1.20	1.20
5	12.16	2.16	2.82	2.82	4.39	1.82	1.82	2.21	2.82	2.82	4.17	0.65	0.65	− 1.22	1.22
6	10.18	0.18	2.50	2.50	4.39	1.00	1.00	2.21	2.50	2.50	4.17	0.51	0.51	− 1.23	1.23
7	8.04	− 1.96	0.04	0.04	4.39	0.00	0.00	2.21	0.04	0.04	4.17	− 0.23	− 0.23	− 1.24	1.24
8	11.46	1.46	1.00	1.00	4.39	0.46	0.46	2.21	1.00	1.00	4.17	0.27	0.27	− 1.24	1.24
9	9.2	− 0.8	0.00	0.00	4.39	0.00	0.00	2.21	0.00	0.00	4.17	− 0.05	− 0.05	− 1.24	1.24
10	10.34	0.34	0.00	0.00	4.39	0.00	0.00	2.21	0.00	0.00	4.17	0.07	0.07	− 1.24	1.24
11	10.03	0.03	0.00	1.03	4.39	0.00	0.53	2.21	0.00	1.03	4.17	0.06	0.51	− 1.24	1.24
12	12.47	2.47	1.97	6.71	4.39	1.47	3.50	2.21	1.97	4.50	4.17	0.78	1.55	− 1.24	1.24
13	11.51	1.51	2.98	10.92	4.39	1.98	5.51	2.21	2.98	7.01	4.17	1.00	1.99	− 1.24	1.24
14	10.4	0.4	2.88	12.70	4.39	1.38	6.41	2.21	2.88	8.41	4.17	0.82	1.96	− 1.24	1.24
15	11.08	1.08	3.46	15.97	4.39	1.46	7.99	2.21	3.46	10.49	4.17	0.90	2.15	− 1.24	1.24
16	10.37	0.37	3.33	17.69	4.39	0.83	8.86	2.21	3.33	11.86	4.17	0.74	2.06	− 1.24	1.24
17	11.62	1.62	4.45	22.32	4.39	1.45	10.98	2.21	4.45	14.48	4.17	1.00	2.38	− 1.24	1.24
18	11.31	1.31	5.29	26.23	4.39	1.76	12.79	2.21	5.26	16.79	4.17	1.10	2.51	− 1.24	1.24
19	9.52	− 0.48	4.31	26.18	4.39	0.28	12.81	2.21	4.28	17.31	4.17	0.62	2.06	− 1.24	1.24
20	11.84	1.84	5.65	31.38	4.39	1.12	15.15	2.21	5.62	20.15	4.17	0.99	2.45	− 1.24	1.24

Table 9 Control charts diagnostic abilities to detect out-of-control signals

Parameters	$\text{ACUSUM}_c^{(1)}$	ACUSUM_E	$\text{AEWMA}_E^{(1)}$	Classical CUSUM
	$\delta_{min}^{(1)+}=1.00$, $\gamma=3.00,\lambda=0.30$	$\delta_{min}^{(1)+}=1.00$, $\gamma=3.00,\lambda=0.30$	$\gamma=3.00$, $\lambda=0.30,L=2.954$	$k=1$
At $\delta=0.00$ first out-of-control signal order	17th	17th	0th	0th
Total out-of-control points or signals	4 out of 20	4 out of 20	0 out of 20	0 out of 20
At $\delta=1.50$ first out-of-control signal order	12th	12th	12th	12th
Total out-of-control points or signals	9 out of 20	9 out of 20	9 out of 20	9 out of 20

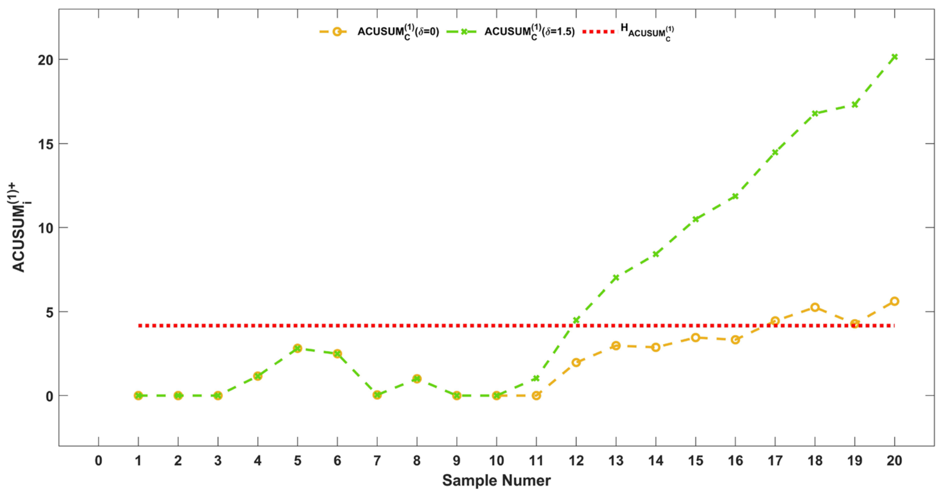
chart against other control charts. To serve this objective, a numerical data of an example from Montgomery [17] are considered. The following parameter combinations which provide the same ARL_0 for control charts are considered: (i) the $\delta_{min}^{(1)+}$ is set equal to 1.00 (e.g., $\delta_{min}^{(1)+}=1.00$), (ii) the λ parameter is considered as 0.30 value, and (ii) γ is chosen between [1.50, 3.00] which is $\gamma=3.00$. To show how the control charts detect different sizes of shift effectively, two different scenarios such as 0 and 1.5 shifts are chosen. There are not any changes (shift) in the first 10 observations and in the last 10 (11th to 20th) observations 1.5 shift is introduced. The numerical results which are given in Table 8 show that the proposed $\text{ACUSUM}_c^{(1)}$ control chart depicts earlier detection ability against the classical CUSUM control at both 0.00 and 1.50 shifts (see Table 9 and Fig. 6a). For example, at $\delta=0.00$, the proposed $\text{ACUSUM}_c^{(1)}$ control chart detects first signal at 17th order of the observations, while the $\text{AEWMA}_E^{(1)}$ and classical CUSUM control chart is unsuccessful to identify any signal (see Table 9 and Fig. 6b, c). However, at $\delta=1.50$, the $\text{ACUSUM}_c^{(1)}$, ACUSUM_E, $\text{AEWMA}_E^{(1)}$, and the classical CUSUM control charts can diagnose signal at 12th order of the observations. Additionally, it can be projected that the proposed $\text{ACUSUM}_c^{(2)}$ control chart also will perform better against the ACUSUM_E, $\text{AEWMA}_E^{(1)}$ and the classical CUSUM control charts because both proposed $\text{ACUSUM}_c^{(1)}$ and $\text{ACUSUM}_c^{(2)}$ performed equally (see Sect. 4).

The findings reveal that the classical CUSUM control chart is normally developed using pre-specific shift; therefore, it may give poor performance when the actual shift is different from the targeted. So, the classical CUSUM control chart cannot provide an overall good detection performance over a range of shifts. In contrast, the proposed $\text{ACUSUM}_c^{(1)}$ control chart procedure uses a location estimator to dynamically change its reference value retains earlier detection ability.

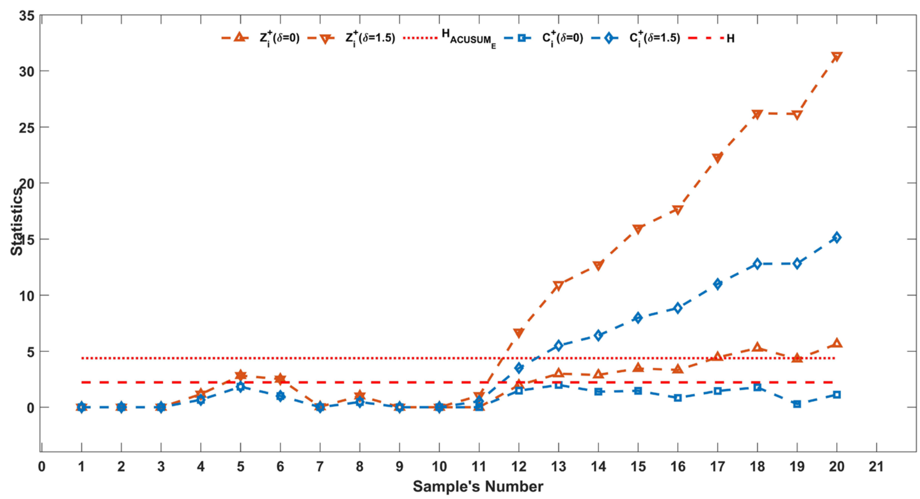
7 Summary, Conclusions, and Recommendations

An adaptive CUSUM (ACUSUM_E) control chart [7] is an advanced form of the classical CUSUM control chart [2]. Its structure is based on the classical EWMA statistic and Huber's function. The ACUSUM_E control chart detects a broad range of shifts in the process location, but the classical EWMA statistic in the ACUSUM_E does not provide explicit rule for parameter choices to diagnose a specific shift as well [4]. To overcome this issue, this study has proposed two ACUSUM control charts, symbolized as ACUSUM_c ($\text{ACUSUM}_c^{(1)}$ and $\text{ACUSUM}_c^{(2)}$) control charts to monitor a specific and a certain range of shift in the process location. The proposed $\text{ACUSUM}_c^{(1)}$ and $\text{ACUSUM}_c^{(2)}$ control charts methodologies are based on the classical CUSUM statistic, generalized likelihood ratio test, and score functions. These techniques help to adjust the reference parameter as a time varying to diagnose shifts effectively. Monte Carlo simulation technique [19] as an algorithm is developed in MATLAB to produce numerical results to obtain performance evaluation measures. Findings based on performance evaluation measures and visual presentation reveal the superiority of the proposed $\text{ACUSUM}_c^{(1)}$ and $\text{ACUSUM}_c^{(2)}$ control charts against other control charts (classical CUSUM, ACUSUM, AEWMA_E, and ACUSUM_E). Besides, to show the implementation procedure for practical point of view, the proposed ACUSUM_C control charts are applied with numerical data to show the significance over other control charts. This study is carried out when process characteristic follows a normal distribution through the simple random sampling scheme. It would be interesting to extend this study for multivariate and for other sampling schemes as well.

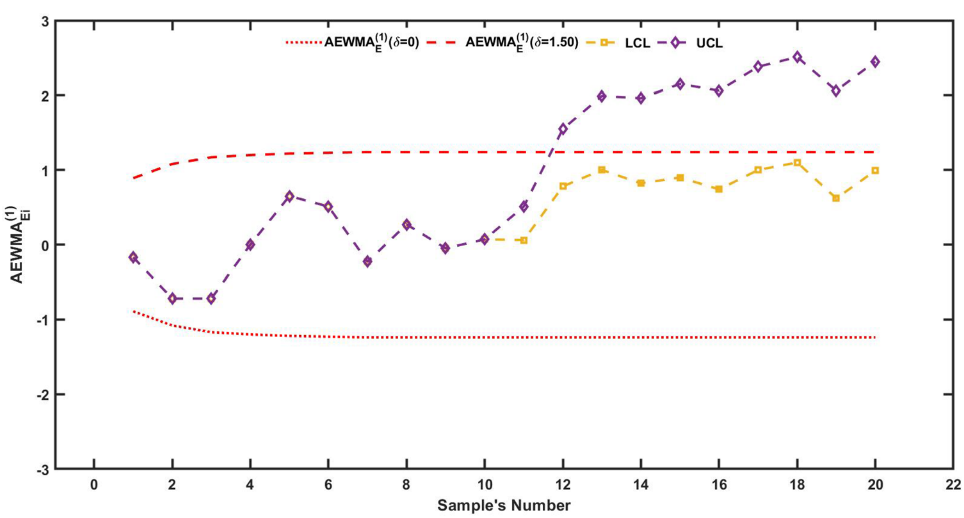
Fig. 6 **a** The graph of $\mathrm{ACUSUM}_c^{(1)}$ control chart at $\delta = 0.00\text{and}1.50$. **b** The graph of $\mathrm{ACUSUM}_c^{(1)}$ and classical CUSUM control charts at $\delta = 0.00\text{and}1.50$. **c** The graph of $\mathrm{AEWMA}_c^{(1)}$ control charts at $\delta = 0.00$ and 1.50



a



b



c

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Appendix

Generalized Likelihood Ratio Test

Let X $(x_i \sim N(\mu_0, \sigma_0^2),\ i = 1, 2, 3, \ldots, n)$ denote a process characteristic of interest that follows a normal probability distribution with known in-control location μ_0 (location parameter) and variance σ_0^2. Assume the process location is in-control and out-of-control is presented as: $H_0 : \mu_0 = 0$ and $H_1 : \mu_0 > \delta_0$ or $\mu_0 < \delta_0$, respectively. The δ_0 presents the process location has been shift at a certain time i_0 $(i < i_0)$. Suppouse f is is the joint density function of x_i. Then, the log-likelihood ratio (LR) of H_0 and H_1 is given as follows:

$$\mathrm{LR}_{i_o} = \ln\left(\frac{f(x_1, x_2, \ldots, x_i|H_1)}{f(x_1, x_2, \ldots, x_i|H_0)}\right)$$

$$\mathrm{LR}_{i_o} = \ln\left(\frac{\prod_{i=1}^{i_o} e^{\left(-x_i^2/2\right)} \times \prod_{i=i_o+1}^{n} e^{\left((x_i-\delta_0)^2/2\right)}}{\prod_{i=1}^{n} e^{\left(-x_i^2/2\right)}}\right)$$

$$\mathrm{LR}_{i_o} = \ln\left(\prod_{i=i_o+1}^{n} e^{\left(-\frac{1}{2}\left((x_i-\delta_0)^2-x_i^2\right)\right)}\right)$$

$$\mathrm{LR}_{i_o} = \sum_{i=i_o+1}^{n} \delta_0\left(x_i - \frac{\delta_0}{2}\right) \quad (1)$$

Equation (1) can be used to detect the changes on the process location and the CUSUM control chart increasement at time n would be

$$I_i = \delta_0\left(x_i - \frac{\delta_0}{2}\right) \quad (2)$$

So, to detect upward and downward shifts in the process location, the one-sided upper and lower CUSUM statistics can be defined as follows:

$$g_i^+ = \max\left[0,\ g_{i-1}^+ + \delta_0\left(x_i - \frac{\delta_0}{2}\right)\right], \quad (3)$$

$$g_i^- = \min\left[0,\ g_{t-1}^- - \delta_0\left(x_i - \frac{\delta_0}{2}\right)\right], \quad (4)$$

respectively. The term δ_0 in Eqs. (3) and (4) is constant and can be absorbed in decision interval. So, the statistics g_i^+ and g_i^+ reduce to the statistics of the classical CUSUM control chart. In practice, the future location shift δ_0 is often unknown and needs to be estimated. In this case, it is intuitive to replace δ_0 by its estimate in the increment I_i. This motivates the use of the linear weight function $w\left(\hat{\delta}_i^+\right) = \hat{\delta}_i^+$.

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